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Trace on $\mathbb{C}_p$

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Let p be a prime number, Q_p the field of p-adic numbers, $\bar{Q}_p$ a fixed algebraic

$F(U, Z)$ if and only if they are conjugate. We view the coefficient of Z in $F(T, Z)$ as the trace of T. Further, we study $F(T, Z)$ viewed as a rigid analytic function and prove that it is defined everywhere on $\mathbb{C}_p$ except on the set of conjugates of $1/T$. The main result (Theorem 7.2) asserts that if $\{T_\alpha\}_\alpha$ is a family of elements of $\mathbb{C}_p$ which are non-conjugate, transcendental over Q_p, and satisfy condition $(*)$ then the functions $\{F(T_\alpha, Z)\}_\alpha$ are algebraically independent over $\mathbb{C}_p(Z)$. In particular, if T is an element of $\mathbb{C}_p$ which satisfies condition $(*)$, then $F(T, Z)$ is transcendental over $\mathbb{C}_p(Z)$ if and only if T is transcendental over Q_p. In proving these results we develop some additional machinery, to be also used in a forthcoming paper which continues the study of orbits of elements in $\mathbb{C}_p$.

INTRODUCTION

Let p be a prime number, Q_p the field of p-adic numbers, $\bar{Q}_p$ a fixed algebraic closure of Q_p, and $\mathbb{C}_p$ the completion of $\bar{Q}_p$ (see [3, 4]). In Section 6 of this paper we construct for elements T of $\mathbb{C}_p$ which satisfy a certain diophantine condition $(*)$ a power series $F(T, Z)$ with coefficients in Q_p and prove that two elements T, U produce the same series $F(T, Z) = F(U, Z)$ if and only if they are conjugate over Q_p. Our motivation for introducing this series $F(T, Z)$ comes from the fact that it controls the trace map on $Q_p[T]$. In particular, when we expand $F(T, Z)$ around zero, we view the coefficient of Z in $F(T, Z)$ as the "trace" of T. Further, we study $F(T, Z)$ as a rigid analytic function. In Theorem 6.1 we prove that for any

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fixed T the function $F(T, Z)$ is a rigid analytic function defined everywhere on $\mathbb{C}_p$ except the set $C(1/T)$ of conjugates of $1/T$. We then study the behavior of $F(T, Z)$ around its set of singularities $C(1/T)$. As a consequence of this investigation we derive Theorem 7.2 which asserts that if $\{T_\alpha\}_\alpha$ is a family of elements of $\mathbb{C}_p$ which are non-conjugate, transcendental over Q_p and satisfy condition $(*)$ then the functions $\{F(T_\alpha, Z)\}_\alpha$ are algebraically independent over $\mathbb{C}_p(Z)$. In particular, if T is an element of $\mathbb{C}_p$ which satisfies condition $(*)$, then $F(T, Z)$ is transcendental over $\mathbb{C}_p(Z)$ if and only if T is transcendental over Q_p. In proving these results we develop some additional machinery, to be also used in further papers concerned with the study of elements and subfields of $\mathbb{C}_p$. The first two sections present some definitions and general results. Here we point out the definition of an interesting class of functions, the so called "equivariant" rigid analytic functions. In Section 3 we show that any $T \in \mathbb{C}_p$ produces a measure μ_T on $\mathbb{C}_p$. This measure is related to the Haar measure on the group $G = \mathrm{Gal}(\bar{Q}_p/Q_p)$ (topologized with the Krull topology). Also, the Haar measure on G induces a (generally unbounded) p-adic measure on $\mathbb{C}_p$ denoted π_T. However, for some elements T of $\mathbb{C}_p$ any Lipschitzian function $f\colon C(T) \to \mathbb{C}_p$ can be integrated with respect to π_T. Then, in Section 6 one defines the "trace function" $F(T, Z)$ and shows that it is a rigid analytic function on the set $\mathbb{C}_p \cup \{\infty\} \setminus C(1/T)$. In Section 7 we state the main results and make some additional comments. An example of an element T for which the function $F(T, Z)$ has infinitely many zeros in any neighborhood of any of its singular points is given. In Section 8 we show that if $\{\alpha_n\}_n$ is a sequence of elements of $\bar{Q}_p$ such that $\lim \alpha_n = T$, then the sequence of measures $\{\mu_{\alpha_n}\}_n$ is weakly convergent to μ_T (Theorem 8.1). This measure μ_T is used in the study of the "metric invariant" $\Delta(T)$ defined in Section 9; this invariant $\Delta(T)$ turns out to be an important tool in understanding the behavior of $F(T, Z)$ around its singular points. In Section 10 we complete the proof of Theorem 7.2.

In a forthcoming paper we shall investigate the structure of the ring of germs of equivariant rigid analytic functions defined in a neighborhood of a given point T in $\mathbb{C}_p$, where a special role is played by our function $F(1/T, Z)$.

1. NOTATIONS, DEFINITIONS, AND RESULTS

1. Let p be a prime number. Denote by Q_p the field of p-adic numbers and by v the p-adic valuation on it (see [3, 7, 8]). Let $\bar{Q}_p$ be a fixed algebraic closure of Q_p and continue to denote by v the unique extension of v to $\bar{Q}_p$. Let $\mathbb{C}_p$ be the completion of $\bar{Q}_p$ with respect to v and let also continue to denote by v the unique extension of v to $\mathbb{C}_p$. For any $x \in \mathbb{C}_p$, denote

$$|x| = \left(\frac{1}{p}\right)^{v(x)}$$

and call $|x|$ *the* (p-adic) *absolute value* of x. If $x \in \bar{Q}_p$ denote $\deg x = [Q_p(x) : Q_p]$. Also denote $\omega(x) = \max(v(x - x')$, where x' runs over all conjugates of x, $x' \neq x$).

2. Let $a \in \mathbb{C}_p$ and r a positive real number. Denote $B[a, r] = \{x \in \mathbb{C}_p, |x - a| \leqslant r\}$ and $B(a, r) = \{x \in \mathbb{C}_p, |x - a| < r\}$. By an *affinoid* (see [8]) we mean a difference set of the form: $U = B[a_0, r_0] \setminus \bigcup_{j=1}^{k} B(a_j, r_j)$, where $a_j \in \mathbb{C}_p$ and r_j are positive real numbers for $0 \leqslant j \leqslant k$.

3. Let $U = B[a_0, r_0] \setminus \bigcup_{j=1}^{k} B(a_j, r_j)$ be an affinoid and let us denote by $O(U)$ the set of all *rigid analytic functions* defined on U with values in $\mathbb{C}_p$. By definition $O(U)$ consists of uniform limits of sequences of rational functions in $C_p(X)$ having no poles in U. If $f \in O(U)$ the *norm of the uniform convergence* on U is defined by

$$\|f\| = \sup_{z \in U} |f(z)|.$$

It is known (see [8, Chap. I]) that any $f \in O(U)$ admits a so called Mittag–Leffler decomposition, i.e., it can be uniquely written as $f = f_0 + \sum_{j=1}^{k} f_j$, where $f_0 \in O(B[a_0, r_0])$ and f_j is a rigid analytic function on the complementary of $B(a_j, r_j)$ (i.e., its poles are in $B(a_j, r_j)$) and "having a zero at infinity," $j = 1, ..., k$. Precisely one has

$$f_0(X) = \sum_{m \geqslant 0} c_{0,m}(X - a_0)^m, \quad |c_{0,m}|\, r_0^m \underset{m \to \infty}{\to} 0$$

$$f_j(X) = \sum_{m \geqslant 1} \frac{c_{j,m}}{(X - a_j)^m}, \quad \frac{|c_{j,m}|}{r_j^m} \to 0, \qquad j = 1, ..., k,$$
$$m \to \infty$$

Moreover one has

$$\|f\| = \sup_{0 \leqslant j \leqslant k} \sup_{\substack{m \geqslant 1 \text{ if } j > 0 \\ m \geqslant 0 \text{ if } j = 0}} |c_{j,m}|\, r_j^{\varepsilon_j m},$$

where

$$\varepsilon_j = \begin{cases} 1 & \text{if } \; j = 0 \\ -1 & \text{if } \; j \neq 0 \end{cases}.$$

4. If $a \in U$ and $f \in O(U)$, then $f(X)$ can be represented around a by a series of the form $f(X) = \sum_{m \geqslant 0} a_m (X-a)^m$ whose radius of convergence $\delta(f, a)$ is "maximal," i.e., one has $\delta(f, a) \geqslant d(a, \mathbb{C}_p \setminus U) =$ the distance from a to the complementary of U.

5. If $f \in O(U)$ and if there exists $a \in U$ and a sequence $\{a_n\}_n$ in U such that $\{a_n\}$ is convergent to a and that $a_n \neq a$, and $f(a_n) = 0$ for any n, then $f(x) = 0$ for any $x \in U$.

This result holds since U has been defined to be a "connected affinoid" (see [7]). However, if the affinoid U is not connected, this result is not true. Indeed, let $U = B[0, 1/2] \cup B[1, 1/2]$. Then the sequence $\{X^{p^n}\}_n$ converges to the zero function on $B[0, 1/2]$ and to the function $f = 1$ on $B[1, 1/2]$.

6. Let $\alpha \in \mathbb{C}_p$ and let δ be a real number. According to [1], by using the pair (α, δ) we can define a valuation $w = w_{(a, \delta)}$ on $\mathbb{C}_p(X)$ and this valuation is a r.t. (residual transcendental) extension of v when δ is a rational number. Precisely, if $P \in \mathbb{C}_p[X]$, and $P = a_0 + a_1(X-\alpha) + \cdots + a_s(X-\alpha)^s$, then $w(P) = \inf_{0 \leqslant i \leqslant s}(v(a_i) + i\delta)$. If δ is rational and $d \in \mathbb{C}_p$ is such that $v(d) = \delta$, denote $t = (\frac{X-\alpha}{d})^*$, the image of $\frac{X-\alpha}{d}$ in k_w, the residue field of w. If k_v is the residue field of v, then $k_w = k_v(t)$, and t is transcendental over k_v. Moreover if δ is not rational then $k_w = k_v$.

For any $f \in \mathbb{C}_p(X)$, denote

$$\|f\|_w = (1/p)^{w(f)}.$$

One has the following result:

PROPOSITION 1.1. *Let $r = (1/p)^\delta$. Then for any $f \in \mathbb{C}_p(X)$ such that $f \in O(B[\alpha, r])$ one has*

$$\|f\|_w = \|f\| = \sup_{z \in B[\alpha, r]} |f(z)|.$$

Proof. Let $Q(X) \in \mathbb{C}_p[X]$ be such that Q has no roots in $B[\alpha, r]$. Then one has: $w(Q(X)) = v(Q(\alpha)) = v(Q(z))$ for any $z \in B[\alpha, r]$. Hence if $f = P/Q \in \mathbb{C}_p(X)$ is such that P and Q have no roots in $B[\alpha, r]$, then $w(f) = v(f(\alpha)) = v(f(z))$ for any $z \in B[\alpha, r]$, and so $\|f\| = \|f\|_w$. Now let us assume $P(X) = \prod_{i=1}^n (X-a_i)$, and let $a_1, ..., a_m$, $1 \leqslant m \leqslant n$ be all the roots of P in $B[\alpha, r]$. Since for any $z \in B[\alpha, r]$ one has $w(X-a_i) = \inf(\delta, v(z-a_i)) \leqslant v(z-a_i)$, it follows that $w(P) \leqslant v(P(z))$. Thus one has: $\|f\|_w \geqslant \|f\|$. First we prove the equality stated in Proposition 1.1 if δ is a rational number. Then it is enough to show that for any $P \in \mathbb{C}_p[X]$, there exists $z_0 \in B[\alpha, r]$

such that $\|P\|_w = |P(z_0)|$. Let $d, q \in \mathbb{C}_p$ be such that $v(d) = \delta$ and that $v(q) = w(P)$. Then one has

$$\frac{P(X)}{q} = \prod_{i=1}^{n} (X - a_i) \frac{1}{q} = \prod_{i=1}^{m} \left(\frac{X-\alpha}{d} + \frac{\alpha - a_i}{d} \right) \prod_{j=m+1}^{n} (X - a_j) \frac{d^m}{q}$$

and so $(P(X)/q)^* = \prod_{i=1}^{m} (t + ((\alpha - a_i)/d)^*) A$, where $A \in k_v$. Since $(P(X)/q)^*$ is a non-zero polynomial in t, there exists $\beta \in \mathbb{C}_p$ such that $v(\beta) = 0$ and that $(P(X)/q)^* (\beta^*) \neq 0$. This means that if in $P(X)/q$ we put $(z_0 - \alpha)/d = \beta$, or equivalently $z_0 = d\beta + \alpha$, then $v(P(z_0)/q) = 0$ and so $v(P(z_0)) = v(q) = w(P)$. To finish we remark that $z_0 \in B[\alpha, r]$, and so $w(f) = v(f(z_0))$, i.e. $\|f\|_w \leqslant \|f\|$, and finally $\|f\|_w = \|f\|$, as claimed.

Now let us assume δ is any real number. Take a decreasing and convergent sequence $\{\delta_n\}_n$ of rational numbers, such that $\delta < \delta_n$ for any n and that $\lim_n \delta_n = \delta$. Denote by w_n the r.t. extension of v to $\mathbb{C}_p(X)$ defined by the pair (α, δ_n) (see [1]). Then it is easy to see that for any $f \in \mathbb{C}_p(X)$ one has $w(f) = \inf_n w_n(f)$. Further, if $r_n = (1/p)^{\delta_n}$, then $\bigcup_n B[\alpha, r_n] = B[\alpha, r]$ and so $O(B[\alpha, r]) = \bigcap_n O(B[\alpha, r_n])$. Let $f \in O(B[\alpha, r])$. Denote by $\|f\|_n$ the norm of f viewed as an element of $O(B[\alpha, r_n])$. Then according to the above considerations, one obtains

$$\|f\| = \sup_n \|f\|_n = \sup_n \|f\|_{w_n} = \|f\|_w.$$

This completes the proof of the proposition.

2. EQUIVARIANT GLOBAL AND LOCAL ANALYTIC FUNCTIONS AROUND $C(T)$

1. An automorphism σ of $\mathbb{C}_p$ over Q_p is said to be continuous if $v(z) = v(\sigma(z))$ for any $z \in \mathbb{C}_p$. Denote by $G = \mathrm{Gal}_c(\mathbb{C}_p/Q_p)$ the group of all continuous automorphisms of $\mathbb{C}_p$ over Q_p. If σ is an automorphism of $\bar{Q}_p/Q_p$ then the extension $\bar{\sigma}$ of σ to $\mathbb{C}_p$ is a continuous automorphism of $\mathbb{C}_p$. Moreover the mapping $\sigma \mapsto \bar{\sigma}$ of $\mathrm{Gal}(\bar{Q}_p/Q_p)$ into $\mathrm{Gal}_c(\mathbb{C}_p/Q_p)$ is an isomorphism. If $T \in \mathbb{C}_p$, let us denote $C(T) = \{\sigma(T), \sigma \in G\}$, the orbit of T. According to [2], $C(T)$ is a compact subset of $\mathbb{C}_p$. If ε is a real number, let us denote $B(C(T), \varepsilon) = \bigcup_{T' \in C(T)} B(T', \varepsilon)$. We shall say that $B(C(T), \varepsilon)$ *is the open ball of radius* ε *around* $C(T)$. Analogously, we consider $B[C(T), \varepsilon]$ *the closed* ball of radius ε around $C(T)$. Since $C(T)$ is compact, $B(C(T), \varepsilon)$ will be a finite union of disjoint open balls.

2. Let $T \in \mathbb{C}_p$. Let ε and r be two real numbers such that $0 < \varepsilon < |T| < r$. (Further we shall let $\varepsilon \to 0$ and $r \to \infty$). Denote

$$U = U_{r,\varepsilon} = B[0, r] \backslash B(C(T), \varepsilon).$$

Since $B(C(T), \varepsilon)$ can be written as a finite disjoint union $B(C(T), \varepsilon) = \bigcup_{j=1}^{N} B(T_j, \varepsilon)$, $T_j \in C(T)$, $1 \leqslant j \leqslant N$, U will be in fact a (connected) affinoid. If $0 < \varepsilon < \varepsilon' < |T| < r$, then $U_{\varepsilon, r} \supset U_{\varepsilon', r}$ and let us denote

$$\bigcup_{0 < \varepsilon < |T|} U_{\varepsilon, r} = B[0, r] \backslash C(T).$$

We shall say that a subset M of $\mathbb{C}_p$ is *equivariant* if for any $\sigma \in G$ one has $\sigma(M) = M$. We remark that subsets like $U_{r,\varepsilon}$ and $B[0, r] \backslash C(T)$ are equivariant sets.

3. Let U be an equivariant subset as above. For any $f \in O(U)$ and any $\sigma \in G$ denote by f^σ the function defined by: $f^\sigma(z) = \sigma^{-1}(f(\sigma(z)))$ for any $z \in U$, $\sigma \in G$. A function $f \in O(U)$ is said to be *equivariant* if $f^\sigma = f$ for any $\sigma \in G$. This means $f(\sigma(z)) = \sigma(f(z))$ for any $z \in U$ and $\sigma \in G$. A rational function $f \in O(U)$ is equivariant if and only if it belongs to $Q_p(X)$. Indeed let $f = P/Q = \sum a_i X^i / \sum b_j X^j \in \mathbb{C}_p(X)$ be an equivariant function. One has $f^\sigma = \sigma^{-1}(f(\sigma(X))) = \sum \sigma(a_i) X^i / \sum \sigma(b_j) X^j = P^\sigma / Q^\sigma$, $\sigma \in G$. Then for any $z \in U$ and any $\sigma \in G$ one has $f(z) = f^\sigma(z)$. This means $P(z)/Q(z) = P^\sigma(z)/Q^\sigma(z)$. Since U is an infinite set, for any $\sigma \in G$ one necessarily has $\sigma(a_i) = a_i$ and $\sigma(b_j) = b_j$ for any i and any j, respectively. Then by the main Theorem of [5] it follows that $a_i, b_j \in Q_p$ for any i and any j. Hence $f \in Q_p(X)$ as claimed. Denote by $A(U)$ the set of all equivariant functions on U. If $f \in A(U)$, then f can be represented in a neighborhood of 0 as a series $f = \sum_{n \geqslant 0} a_n z^n$ whose coefficients a_n belong to Q_p. Also, if $f = f_0 + \sum_{j=1}^{N} f_j$ is the Mittag–Leffler decomposition of f, then the coefficients of f_0 are in Q_p and the coefficients of the functions f_j, $1 \leqslant j \leqslant N$ verify some relations which follow easily from the fact that f is equivariant. For a large class of subsets U of $\mathbb{C}_p$ the structure of equivariant analytic functions on U can be easily described:

THEOREM 2.1. *Let U be an equivariant subset of $\mathbb{C}_p$ of the form $U = \bigcup_{\sigma \in G} \sigma B$, where B is a closed ball. An element $f \in O(U)$ belongs to $A(U)$ if and only if it is a uniform limit of functions from $Q_p(X)$ without poles in U.*

Proof. Denote by $A'(U)$ the subset of $A(U)$ consisting of all the elements f such that f is the limit of a sequence of elements of $Q_p(X)$ without poles in U. We must show that $A(U) = A'(U)$. Denote by K' the quotient field of $A'(U)$ and extend the norm $\| \ \|$ to K' in a suitable manner.

Let K be the completion of K' with respect to the norm $\| \ \|$ and continue to denote by the same symbol the norm of K. Furthermore denote by $\bar{K}$ a fixed algebraic closure of K and extend (uniquely) the norm of K to $\bar{K}$. Then by $\tilde{\bar{K}}$ denote the completion of $\bar{K}$. Let $f \in A(U)$. For any $\varepsilon > 0$ there exists $g \in \bar{Q}_p(X)$ such that $g \in O(U)$ and $\|f-g\| < \varepsilon$. Then for any $\sigma \in G$ one has: $\|g_\sigma - g\| < \varepsilon$, where $g^\sigma(z) = \sigma^{-1}(g(\sigma(z)))$ for any $z \in U$. It is clear that g is integral over $A'(U)$ since the set $\{g^\sigma\}_{\sigma \in G}$ contains all the conjugates of g over K. Then by the main result of [5] it follows that there exists $h \in K$ such that $\|f-h\| < \varepsilon p^{1/(p-1)^2}$, and by the proof of this result h is also integral over $A'(U)$. Since ε was arbitrary, one has $f \in K$. Therefore in order to prove that $f \in A'(U)$ it is enough to show that $A'(U)$ is integrally closed in K. For any $a \in U$ denote by $r(a)$ the greatest real number r such that $B[a, r] \subseteq U$. We remark that $A'(U) = \bigcap_{a \in U} A'(B[a, r(a)])$. Denote by K_a the completion of the quotient field of $A'(B[a, r(a)])$. In order to prove that $A'(U)$ is integrally closed in K its is enough to show that $A'(B[a, r(a)])$ is integrally closed in K_a. Let $r = r(a)$ and $\delta = \log_{1/p} r$. Denote $w = w_{(a, \delta)}$, the valuation on $\mathbb{C}_p(X)$ defined by the pair (a, δ) (see [1]). If $f \in A(U)$ then $f \in O(B[a, r])$ and so according to Proposition 1.1 one has: $\|f\|_1 = \|f\|_w$, where $\|f\|_1$ denotes the norm of f viewed as an element of $O(B[a, r])$. Moreover $A'(B[a, r])$ is just $\widetilde{Q_p[X]}$, the completion of $Q_p[X]$ with respect to the restriction of w to $Q_p(X)$. According to the proof of Proposition 1.1, we may assume that δ is a rational number. Let $b \in \bar{Q}_p$ be such that $v(b) = \delta$ and let $L = Q_p(a, b) = Q_p(y)$. To show that $A'(B[a, r])$ is integrally closed in K_a, it is enough to show that $\widetilde{L[X]}$ is integrally closed in $\widetilde{L(X)}$, where the completion is made with respect to the restriction of w to $L(X)$, denoted by w_0. Let us denote $Y = \frac{X-a}{b}$. Then w_0 is just the Gauss-valuation on $L(Y)$ (see [1]). If k is the residue field of L (with respect to v), and if we denote $t = Y^*$, the image of Y into k_{w_0}, the residue field of $L(Y)$ with respect to w_0, then one has $k_{w_0} = k(t)$. Moreover the residue ring of $L[Y]$ is just $k[t]$. Let S_1 be a system of representatives for $k[t]$ in $L[Y]$. Denote $S = \{s/s', s, s' \in S, s' \neq 0\}$.Then S is a system of representatives of $k(t)$. If π is a uniformizing element of the field L, then any $f \in \widetilde{L(Y)}$ such that $w_0(f) \geqslant 0$ can be represented uniquely in the form: $f = \sum_{i=0}^{\infty} s_i \pi^i$, where $s_i \in S$, for all $i \geqslant 0$. Now if $f \in \widetilde{L(X)}$ is integral over $\widetilde{L[Y]}$, then $w_0(f) \geqslant 0$. It is clear that f^*, the image of f in the residue of w_0 is integral over $k[t]$, and so belongs to it. Hence one has $f^* = s_0^*$, where $s_0 \in S_1$. If $f_1 = (f - s_0)/\pi$, then $w_0(f_1) \geqslant 0$ and f is also integral over $\widetilde{L[Y]}$. Then $f_A^* = s_1^*$, where $s_1 \in S_1$, and so on. Finally, one obtains $f \in \widetilde{L[Y]}$.

So far we know that if $f \in A(B[a, r])$, then f is a (uniform) limit of rational functions over the field $Q_p(y)$, for $y \in \bar{Q}_p$, without poles in $B[a, r]$. Hence f can be uniquely represented as a sum: $f = f_0 + yf_1 + \cdots + y^{s-1} f_s$, where $s = \deg y$, and $f_i \in A'(B[a, r])$, $0 \leqslant i \leqslant s-1$. But for any $\sigma \in G$ one has: $f = f_\sigma = f_0 + \sigma(y) f_1 + \cdots + \sigma(y)^{s-1} f_{s-1}$. If $y \notin Q_p$, these equalities hold true if and only if $f_i = 0$, if $i > 0$. Finally one necessarily has $f \in A'(B[a, r])$, as claimed.

4. By an equivariant rigid analytic function on $B[0, r] \backslash C(T) = \bigcup_{0 < \varepsilon < |T|} U_{\varepsilon, r}$ we understand a function f such that for any $0 < \varepsilon < |T|$ the restriction of f to $U_{\varepsilon, r}$ belongs to $A(U_{\varepsilon, r})$. Let us denote by $A(B[0, r] \backslash C(T))$ the set of all equivariant rigid analytic functions on $B[0, r] \backslash C(T)$. We remark that the norm $\|f\|$ of such a function f is not necessary finite on $B[0, r] \backslash C(T)$.

5. If $|T| < r_1 < r_2$, then $B[0, r_1] \backslash C(T) \subset B[0, r_2] \backslash C(T)$ and $\bigcup_{r > |T|} (B[0, r] \backslash C(T)) = \mathbb{C}_p \backslash C(T)$. Let us denote by $A(\mathbb{C}_p \backslash C(T))$ the set of all functions f such that the restriction of f to $B[0, r] \backslash C(T)$ belongs to $A(B[0, r] \backslash C(T))$, for any $r > |T|$. Also, by $A(\mathbb{C}_p \cup \{\infty\} \backslash C(T))$ we denote the subset of $A(\mathbb{C}_p \backslash C(T))$ consisting of all functions f which are analytic at infinity (i.e., if we change z by $1/z$, the new function is analytic around zero). Let $0 < \varepsilon_2 < \varepsilon_1 < |T| < r_1 < r_2$ be real numbers, and let us denote $U_i = B[0, r_i] \backslash B(C(T), \varepsilon_i)$, $i = 1, 2$. If $f \in A(\mathbb{C}_p \backslash C(T))$ denote by f_i the restriction of f to U_i, $i = 1, 2$. Let

$$f_1 = f_{0,1} + \sum_{j=1}^{N_1} f_{j,1}, \qquad f_2 = f_{0,2} + \sum_{j=1}^{N_1} \sum_{l=1}^{l_j} f_{j,2}$$

be the corresponding Mittag–Leffler decompositions of f_1 and f_2. The form of the decomposition of f_2 comes from the fact that every $B(T_j, \varepsilon_1)$ is a disjoint union of l_j balls of radius ε_2. By hypothesis one has $U_1 \subset U_2$, and so if we look at the restriction of f_2 to U_1 one obtains: $f_{0,2 \mid U_1} = f_{0,1}$ and $\sum_{l=1}^{l_j} f_{j,l \mid U_1} = f_{j,1}$, $j = 1, ..., N$. In conclusion, any $f \in A(\mathbb{C}_p \backslash C(T))$ has a component f_0 which can be extended to $\mathbb{C}_p$; i.e., it is an entire function. Moreover, if $f \in A(\mathbb{C}_p \cup \{\infty\} \backslash C(T))$ then f_0 can be extended analytically to $\mathbb{C}_p \cup \{\infty\}$ and so by Liouville's Theorem it is constant. If $f(\infty) = 0$ (i.e. f vanishes at infinity) then $f_0(\infty) = 0$ and so $f_0 = 0$.

6. Let $M \subset \mathbb{C}_p$ be a set, and let us denote $U_M = \mathbb{C}_p \cup \{\infty\} \backslash \bigcup_{\sigma \in G} \sigma(M)$. Let us assume that M if finite modulo G (i.e. there exist only finitely many elements in M, say $\{T_1, ..., T_k\}$, which are not conjugate). Then any $f \in A(U_M)$ can be written as $f = \sum_{j=1}^{k} f_j$, where $f_j = A(U_{T_j})$. However such a decomposition is not necessary unique. Let us denote $A_0(U_M) = \{f \in A(U_M), f(\infty) = 0\}$. Then $A(U_M) = \mathbb{C}_p \oplus A_0(U_M)$ and $A_0(U_M) = \bigoplus_j A_0(U_{T_j})$.

7. Let us consider a (not necessary connected) affinoid $B[C(T), \varepsilon]$ and denote by $A(B[C(T), \varepsilon])$ the set of all equivariant rigid analytic functions on $B[C(T), \varepsilon]$. A natural question is to see what happens when $\varepsilon \to 0$, i.e., to study the "germs" of such functions around $C(T)$. Denote

$$O_T = \varinjlim_{\varepsilon \to 0} A(B[C(T), \varepsilon]) = \bigcup_{\varepsilon > 0} A(B[C(T), \varepsilon])$$

the *ring of germs of equivariant rigid analytic functions* around *T*.

Remark 2.2. T is transcendental over Q_p if and only if O_T is a field.

Proof. Let T be transcendental and $f \in O_T$, $f \neq 0$. There exists $\varepsilon > 0$ such that $f(z) \neq 0$ for any $z \in B[C(T), \varepsilon]$. Indeed, otherwise there will be a sequence $\{\varepsilon_n\}_n$ of strictly positive real numbers convergent to 0, and for any n an element $z_n \in B[C(T), \varepsilon_n]$ such that $f(z_n) = 0$. Since the sequence $\{z_n\}_n$ tends to an element of $C(T)$, it will follow that $f(T) = 0$, since f is equivariant. We now prove that this gives a contradiction. For, let ε' be small enough such that $f \in A(B[C(T), \varepsilon'])$. If $\alpha \in B[C(T), \varepsilon] \cap \bar{Q}_p$ then for any $z \in B[C(T), \varepsilon]$ we get $f(z) = \sum_{n \geqslant 0} b_n(z-\alpha)^n$, $b_n \in Q_p(\alpha)$ for all n. By the general theory of affinoid algebras (see [7, Chap. 1]) it follows that f has a finite number of zeros in $B[C(T), \varepsilon']$, and all belong to $\bar{Q}_p$. Then for a suitable $\varepsilon' \leqslant \varepsilon$ one has $f(z') \neq 0$ if $z' \in B(z, \varepsilon')$. This means that $f(x) \neq 0$ for any $x \in B(C(T), \varepsilon')$ and so $1/f \in A(B[C(T), \varepsilon']) \subset O_T$. Moreover if T is algebraic over Q_p then the minimal polynomial f of T over Q_p cannot be invertible in $A(B[C(T), \varepsilon])$ for any ε.

8. Besides the "global" (equivariant) rigid analytic functions around $C(T)$ it is interesting to consider also the "local" (equivariant) rigid analytic functions around $C(T)$. These are functions f: $B[C(T), \varepsilon] \setminus C(T) \to \mathbb{C}_p$ which are equivariant rigid analytic. Denote by $A(B[C(T), \varepsilon] \setminus C(T))$ the set of such functions. If $\varepsilon > \varepsilon'$, then one has an inclusion (given by the restriction) $A(B[C(T), \varepsilon] \setminus C(T)) \to A(B[C(T), \varepsilon'] \setminus C(T))$. Hence we may consider the ring $G_T = \bigcup_{\varepsilon > 0} A(B[C(T), \varepsilon] \setminus C(T))$ of germs of equivariant rigid analytic functions around $C(T)$, but not necessarily on $C(T)$. It is clear that one has $O_T \subset G_T$. If T is algebraic over Q_p then the minimal polynomial of T has an inverse in G_T.

Remark 2.3. Let T be transcendental over Q_p. Denote by $\widetilde{Q_p(T)}$ the completion of $Q_p(T)$ in $\mathbb{C}_p$. The mapping $f \mapsto f(T)$ defines O_T as a subfield of $\widetilde{Q_p(T)}$.

Proof. Let $f \in A(B[C(T), \varepsilon])$. Since f is equivariant one gets $\sigma(f(T)) = f(T)$ for any $\sigma \in H(T) = \{\sigma \in G \mid \sigma(T) = T\}$. Then by the main result of [5], it follows that $f(T) \in \widetilde{Q_p(T)} = \widetilde{Q_p[T]}$.

Remark 2.4. Let $f \in G_T$. Assume that the (finite) limit $l = \lim_{z \to T} f(z)$ exists. Then $l \in \widetilde{Q_p(T)}$.

Proof. In the proof we shall use Proposition 5.3. Since $\widetilde{Q_p(T)} \cap \bar{O}_T$ is dense in $\widetilde{Q_p(T)}$ (see [2, Theorem 1.1]), there exists a convergent sequence $\{z_n\}_n \subseteq \widetilde{Q_p(T)} \cap \bar{O}_p$, $\lim z_n = T$. For any n we can find a real number $\varepsilon_n > 0$ such that the sequence $\{\varepsilon_n\}$ is convergent to zero, and that $z_n \in B[C(T), \varepsilon_n]$. Moreover f is analytic around z_n. If $\sigma \in G$ is such that $\sigma(T) = T$ (i.e. $\sigma \in H(T)$) then also $\sigma(z_n) = z_n$ and $\sigma f(z_n) = f(z_n)$ for any n. Since $\lim f(z_n) = l$ one has $\sigma(l) = l$ for all $\sigma \in H(T)$, i.e., $l \in \widetilde{Q_p(T)}$, (see Proposition 5.3) as claimed.

We also remark that it can be proved that the above mapping $f \mapsto f(T)$ is not surjective.

3. THE HAAR MEASURE ON THE GROUP G

1. Consider the group $G = \mathrm{Gal}(\bar{Q}_p/Q_p)$, endowed as usually with the Krull topology (see [4, p. 104]). Then G is a compact group and so it carries a canonical Haar measure μ normalized so that $\mu(G) = 1$. Also, G acts continuously on $\mathbb{C}_p$. If H is an open subgroup of G, then $[G : H] < \infty$ and one has

$$\mu(H) = \frac{1}{[G : H]}.$$

Since the open subgroups of G generate the clan of μ-measurable sets of G, it follows by the above equality that any measurable set M of G can be decomposed as a (generally infinite) union of disjoint cosets of open subgroups: $M = \bigcup_{i=1}^{\infty} B_i$. Then one has $\mu(M) = \sum_i \mu(B_i)$. Since $\mu(B_i)$ is a rational number, let us denote by $\pi(B_i)$ the same number viewed as a p-adic number. Denote by $\pi(M)$ the p-adic number (finite or infinite) defined by the sum $\sum_{i=1}^{\infty} \pi(B_i)$. In this way we get a "$p$-adic" measure on G, usually called the *p-adic Haar measure*. Although π is not a "measure" in the usual sense, some functions f: $G \to \mathbb{C}_p$ can be integrated with respect to π.

Remark 3.1. Let $T \in \mathbb{C}_p$. Denote $\varphi = \varphi_T$: $G \to C(T)$ the mapping $\varphi(\sigma) = \sigma(T)$, for all $\sigma \in G$. The set $C(T)$ is naturally endowed with the topology induced from $\mathbb{C}_p$. Then the mapping φ is continuous and closed, i.e., the topology of $C(T)$ induced by $\mathbb{C}_p$ is the same as the quotient topology induced by the surjective mapping φ.

Proof. Let φ be a positive real number, $a \in \mathbb{C}_p$, and let $V = C(T) \cap B(a, \varepsilon)$; then $\varphi^{-1}(V)$ is open in G. For, let us assume $a \in \bar{Q}_p$, and let $\sigma(T) \in V$. Then $\sigma \in \varphi^{-1}(V)$. Consider the open set of G: $V(\sigma, \sigma^{-1}(a)) = \{\tau \in G, \tau(\sigma^{-1}(a)) = a\}$.

If $\tau \in V(a, \sigma^{-1}(a))$, then one has $|a-\tau(T)| = |\tau(\sigma^{-1}(a)) - \tau(T)| = |\sigma^{-1}(a) - T| = |a-\sigma(T)| < \varepsilon$. Hence $V(\sigma, \sigma^{-1}(a)) \subseteq V$, and so V is open in G. To show that φ is closed let F be a closed subset of G and let $y \in \overline{\sigma(F)}$ be the closure of $\varphi(F)$ in $\mathbb{C}_p$. Since $C(T)$ is closed in $\mathbb{C}_p$ (see [2, Proof of Theorem 3.5]), then $y \in C(T)$ and so $y = \sigma(T)$, for $\sigma \in G$. Let $\{\sigma_n(T)\}_n$ be a sequence of elements of $\varphi(F)$ which converges to y. If σ' is a limit point of the subset $\{\sigma_n\}_n$ of F, then $\sigma' \in F$, since it is closed in G, and since φ is continuous, $\varphi(\sigma') = \sigma'(T) = \sigma(T) \in \varphi(F)$. Hence $\varphi(F)$ is closed.

Let T be an element of $\mathbb{C}_p$ and $H = H(F) = \{\sigma \in G / \sigma(T) = T\}$. Then by the previous Remark the quotient space G/H can be homeomorphically identified with $C(T)$. Let $B(a, \varepsilon)$ be an open ball in C_p so that $B(a, \varepsilon) \cap C(T) = F \neq \emptyset$. Then F is a compact set and let us define

$$\mu_T(T(a, \varepsilon)) = \mu(\varphi^{-1}(F)).$$

According to [9, Theorem A, p. 253], this equality defines the Haar measure induced by μ on the set $C(T)$.

Remark 3.2. The measure μ_T depends only on the closed subgroup $H = H(T)$ and not on T.

Denote by $\mathscr{B}$ the Borelian clan on $\mathbb{C}_p$ generated by the open balls $B(a, \varepsilon)$ with $a \in \mathbb{C}_p$ and $\varepsilon > 0$. Denote by $N(T, \varepsilon)$ the number of distinct ε-balls which intersect $C(T)$. Then for any ε-ball $B(a, \varepsilon)$, one defines

$$\mu_T(B(a, \varepsilon)) = \mu_T(B(a, \varepsilon) \cap C(T)).$$

It is easy to see that one has

$$\mu_T(B(a, \varepsilon)) = \begin{cases} \dfrac{1}{N(T, \varepsilon)} & \text{if} \quad B(a, \varepsilon) \cap C(T) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that μ_T defines a measure on $\mathscr{B}$. If T, T' are two elements of $\mathbb{C}_p$ such that $H(T) = H(T')$, then in general μ_T and $\mu_{T'}$ are distinct measures on $\mathscr{B}$.

2. By a *p-adic* measure on $\mathbb{C}_p$ we shall mean a map $\lambda \colon \mathscr{B} \to \mathbb{C}_p$ such that $\lambda(\bigcup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} \lambda(B_i)$ whenever $B_1, B_2, \ldots$ are mutually disjoint elements of $\mathscr{B}$. We remark that the sum in the right-hand side of the last equality is not necessary finite. For any ball $B(a, \varepsilon)$ the number $\mu_T(B(a, \varepsilon))$ defined above is a rational number. Let us denote by $\pi_T(B(a, \varepsilon))$ the rational number $\mu_T(B(a, \varepsilon))$ viewed as a p-adic number. For any $B \in \mathscr{B}$, denote by $\pi_T(B)$ the corresponding (finite or infinite) sum of p-adic numbers. In this way we get a p-adic measure $\pi_T \colon \mathscr{B} \to \mathbb{C}_p$.

Remark 3.3. For any $x \in C(T)$, let δ_x be the p-adic Dirac measure centered at the point x. Let $\{x_n\}_n$ be a sequence of elements of $\mathbb{C}_p$ such that $\lim \lambda_n = 0$. Denote $\lambda = \sum \lambda_n \delta_{x_n}$. Then λ is a p-adic measure on $C(T)$ and any p-adic measure on $C(T)$ is of this form. The proof follows as in [13, p. 275].

A p-adic measure λ on $\mathbb{C}_p$ is said to be *bounded* if there exists a real number M such that for any $B \in \mathcal{B}$ one has: $|\lambda(B)| \leqslant M$.

An extension $Q_p \subseteq K \subseteq \bar{Q}_p$ is called *p.b.d.* if there exists a natural number s such that p^s does not divide $[L : Q_p]$ for any finite subextension $Q_p \subseteq L \subseteq K$. Accordingly, the element $T \in C_p$ will be called a *p.b.d.*-element if the algebraic extension $\widetilde{Q_p(T)} \cap \bar{Q}_p$ is a p.b.d. extension. One has the following result:

Remark 3.4. If T is a p.b.d. element then the p-adic measure π_T is bounded.

Proof. Let $\{\alpha_n\}_n$ be a sequence of elements of $\widetilde{Q_p(T)} \cap \bar{Q}_p$ such that $T = \lim_n \alpha_n$. Let s be a natural number such that $p^s \nmid d_n$, for any n, where $d_n = \deg \alpha_n$. If $B(a, \varepsilon)$ is an ε-ball, then one has $\mu_T(B(a, \varepsilon)) = \mu_{\alpha_n}(B(a, \varepsilon))$ for n big enough. Hence, by the definition of π_T one has: $|\pi_T(B(a, \varepsilon))| \leqslant p^s$. This shows that π_T is bounded since any measurable set is a countable union of disjoint open balls.

Example 3.5. Let $p = 2$. We shall use the results of [2, Sect. 3] to define an element T of $\mathbb{C}_2$ such that the p-adic measure π_T is not bounded.

For any integer $n \geqslant 0$ we shall find a polynomial $f_n \in Q_2[X]$, such that:

(1) f_n is irreducible of degree 2^n. If α is a root of f_n then the extension $Q_2(\alpha)/Q_2$ is unramified.

(2) Let $(\alpha_{n,i})_{1 \leqslant i \leqslant 2^n}$ be all the roots of f_n. Then one has $v(\alpha_{n+1,2i} - \alpha_{n,i}) = n = v(\alpha_{n+1,2i-1} - \alpha_{n,i})$.

(3) If $n \geqslant 2$, then $v(\alpha_{n,i} - \alpha_{n,j}) < n$ if $|i - j| > 1$.

Let us define $f_0 = X - 1$ and let us assume that $n \geqslant 0$ and that f_n has been defined such that the conditions (1)–(3) are accomplished. Let k_n be the residue field of $Q_2(\alpha)$, where α is a root of f_n, and let G_{n+1} be an irreducible polynomial of degree two over k_n. Denote by w the residual transcendental extension of v to $Q_2(X)$ defined by the minimal pair $(\alpha, n+1)$. (see [1]). We remark that the inductive hypothesis enables us to check that w is independent of α (see also [12]). Let f_{n+1} be a lifting of G_{n+1} with respect to w (see [12]). Then f_{n+1} is irreducible and of degree 2^{n+1}. Moreover, its roots $\{\alpha_{n+1,i}\}_i$ can be labeled such that the above conditions are verified. By the above conditions (1)–(3) it follows easily that the sequence $\{\alpha_n\}_{n \geqslant 0}$ where $\alpha_n = \alpha_{n,1,n \geqslant 0}$, is convergent and let

$T = \lim \alpha_n$. Since the sequence $(\alpha_n)_{n \geqslant 0}$ is distinguished (see [2]), T will be a transcendental element over Q_2. Also, by conditions (1)–(3) it follows that for any n and any natural number m, the number $N(1/2^m)$ of balls of radius $1/2^m$ which covers $C(\alpha_n)$ is defined by

$$N(T, 1/2^m) = \begin{cases} 2^m & \text{if} \quad m \leqslant n \\ 2^n & \text{if} \quad m > n. \end{cases}$$

By these considerations and Proposition 3.6 it follows that the 2-adic measure π_T is not bounded. This example will also be considered further in Section 8.

3. At this point we prove some results which will be used frequently in what follows.

PROPOSITION 3.6. *Let $T_1, T_2 \in \mathbb{C}_p$ be such that $|T_1 - T_2| < \varepsilon$. Then for any ε-ball $B(a, \varepsilon)$ one has: $\mu_{T_1}(B(a, \varepsilon)) = \mu_{T_2}(B(a, \varepsilon))$.*

Proof. Let $B(a', \varepsilon)$ be such that $B(a', \varepsilon) \cap C(T_1) \neq \emptyset$. If $\sigma \in G$ and $\sigma(T_1) \in B(a', \varepsilon)$, then $|a' - \sigma(T_2)| = |a' - \sigma(T_1) + \sigma(T_1) - \sigma(T_2)| \leqslant \sup(|a' - \sigma(T_1)|, |T_1 - T_2|) < \varepsilon$. Hence $B(a', \varepsilon) \cap C(T_2) \neq \emptyset$. In conclusion $N(T_1, \varepsilon) = N(T_2, \varepsilon)$.

It is clear that if α is algebraic over Q_p, then for any $\varepsilon > 0$ the number $N(\alpha, \varepsilon)$ divides $\deg \alpha$. Hence by Proposition 3.6 it follows:

COROLLARY 3.7. *If $t \in \mathbb{C}_p$ and $\alpha \in \bar{Q}_p$ is such that $|T - \alpha| < \varepsilon$, then $(\deg \alpha)\, \mu_T(B(a, \varepsilon))$ is a natural number for any $a \in \mathbb{C}_p$.*

For any $\delta > 0$, let us denote $C(T, \delta) = B[T, \varepsilon] - B(T, \varepsilon) = \{z \in \mathbb{C}_p, |T - z| = \delta\}$. The set $C(T, \delta)$ is a disjoint union of δ-balls. Hence, if $\delta \geqslant \varepsilon$ then $(\deg \alpha)(\mu_T(C(T, \delta)))$ is necessarily a natural number.

Remark 3.8. Let $T \in \mathbb{C}_p$ and let $\varepsilon > 0$ be a real number. With the notations in Section 2, one can see that $B(C(T), \varepsilon)$ is the union of $N(T, \varepsilon)$ balls, any two disjoint. Now we remark that for $\varepsilon_1 < \varepsilon_2$ the number $N(T, \varepsilon_2)$ divides $N(T, \varepsilon_1)$.

Indeed, the result is obviously true if T is algebraic over Q_p. If T is not algebraic over Q_p the result follows from Proposition 3.6.

4. INTEGRATION WITH RESPECT TO THE p-ADIC HAAR MEASURE π_T

1. Denote by $\mathscr{C}_p = \mathscr{C}_p(C(T))$ the set of all continuous functions defined on $C(T)$ with values in $\mathbb{C}_p$. If $f \in \mathscr{C}_p$ define $\|f\|_T = \sup_{x \in C(T)} |f(x)|$.

It is easy to see that $\mathscr{C}_p$ is a Banach algebra over $\mathbb{C}_p$ (see [7]). A function $f: C(T) \to \mathbb{C}_p$ is called *locally constant* if for any $x \in C(T)$ there exists a ball $B(x, \varepsilon)$ such that $f(x) = f(y)$ for all $y \in B(x, \varepsilon)$. Since $C(T)$ is compact, the real number ε can be chosen to be the same for all $x \in C(T)$. Then we say that f is an ε-*locally constant* function. Any locally constant function is continuous. Denote by $\mathscr{E}_p$ the subset of $\mathscr{C}_p$ consisting of all locally constant functions. It is clear that $\mathscr{E}_p$ is dense in $\mathscr{C}_p$. By an ε-*ball on* $C(T)$ we mean a set $D(x, \varepsilon) = B(x, \varepsilon) \cap C(T)$, where $x \in C(T)$. For any positive real number ε, the compact set $C(T)$ is uniquely covered by a set $\{D(a_i, \varepsilon)\}_{1 \leqslant i \leqslant N(T, \varepsilon)}$ of ε-balls on $C(T)$, any two disjoint. We shall say that $\{D(a_i, \varepsilon)\}_i$ is the ε-covering of $C(T)$. Let f be a locally constant function and let $\{\varepsilon_n\}_n$ be a sequence of real numbers such that $\lim_n \varepsilon_n = 0$. For any n, let us denote

$$\Phi(f, \varepsilon_n) = \sum_{i=1}^{N(T, \varepsilon_n)} f(a_i)\, \pi_T(D_i(a_i, \varepsilon_n)).$$

It is clear that the sequence $\{\Phi(f, \varepsilon_n)\}_n$ is ultimately constant. Denote by $\Phi(f)$ the limit of this sequence. By the above considerations (see Remark 3.8) it follows that the number $\Phi(f)$ does not depend on the sequence $\{\varepsilon_n\}$. Let us denote

$$\Phi(f) = \int_{C(T)} f \, d\pi_T.$$

It is easy to see that $\mathscr{E}_p$ is a $\mathbb{C}_p$-subspace of $\mathscr{C}_p$ and the mapping $\Phi: \mathscr{E}_p \to \mathbb{C}_p$ is a functional (linear and continuous). Since $\mathscr{E}_p$ is dense in $\mathscr{C}_p$, a natural question is whether Φ can be extended to $\mathscr{C}_p$. Precisely, if $f \in \mathscr{C}_p$, and $f = \lim f_n$, $f_n \in \mathscr{E}_p$ for all n, then we ask whether the element $\bar{\Phi}(f) = \lim_n \Phi(f_n)$ is defined and is independent of the sequence $\{f_n\}_n$. If $\bar{\Phi}(f)$ is defined we shall say that f is integrable on $C(T)$ with respect to the p-adic measure π_T. Then the number

$$\bar{\Phi}(f) = \int_{C(T)} f \, d\pi_T$$

will be called the "integral" of f on $C(T)$.

Remark 4.1. (a) If π_T is a bounded p-adic measure then any element of $\mathscr{C}_p$ is integrable on $C(T)$. This is true if for example T is a p.b.d. element (see Remark 3.4).

(b) Let $f \in \mathscr{C}_p$. For any ε-covering $\{D(a_i, \varepsilon)\}_i$ of $C(T)$ let us denote

$$\Phi(f, a_i, \varepsilon) = \sum_i f(a_i)\, \pi_T(D(a_i, \varepsilon))$$

(*the Riemannian sum* associated to f, a_i, and ε). It is plain to see that f is integrable on $C(T)$ if the set of elements $\{\Phi(f, a_i, \varepsilon)\}_{a_i, \varepsilon}$ has a (finite) unique limit point.

2. At this point we discuss the existence of integrable functions on $C(T)$ with respect to the measure π_T.

We shall say that an element $T \in \mathbb{C}_p$ is *Lipschitzian* if one has

$$\lim_{\varepsilon \to 0} \frac{\varepsilon}{|N(T, \varepsilon)|} = 0.$$

Here $|N(T, \varepsilon)|$ stands for the p-adic absolute value of the integer $N(T, \varepsilon)$. Any element $T \in \bar{Q}_p$ is Lipschitzian. For a sequence $\{\alpha_n\}_n$ of elements in $\bar{Q}_p$ we introduce the condition

$$\frac{|\alpha_{n+1} - \alpha_n|}{\inf(|d_n|, |d_{n+1}|)} \to 0, \tag{$*$}$$

where $d_n = \deg \alpha_n$. Note that any sequence which verifies condition $(*)$ is convergent in $\mathbb{C}_p$. We say that an element $T \in \mathbb{C}_p$ *satisfies the property* $(*)$ or is a $(*)$-*element* if it is the limit of a sequence $\{\alpha_n\}_n$ in $\bar{Q}_p$ which verifies condition $(*)$.

Remark 4.2. (a) By Proposition 3.6 it follows that any $(*)$-element is Lipschitzian.

(b) According to [2, Proposition 5.2] any closed subfield F of $\mathbb{C}_p$ is of the form $\widetilde{Q_p(T)}$, where T is an $(*)$-element, hence Lipschitzian.

(c) If T is a $(*)$-element, then any element $T \in Q_p[T]$ is also a $(*)$-element.

The following example shows that not all elements of $\mathbb{C}_p$ are Lipschitzian.

EXAMPLE 4.3. Let $p \geqslant 3$ be a prime number and let ε_n be a primitive root of degree p^n of unity. Denote $K = \bigcup_{n=1}^{\infty} Q_p(\varepsilon)$. We remark that $Q_p(\varepsilon_n)$ is the unique extension of $Q_p(\varepsilon_1)$ contained in K and having the degree p^{n-1}. Let us denote $\alpha_n = \varepsilon_2 + p\varepsilon_q + \cdots + p^{n-1}\varepsilon_n$ and let $T = \lim_n \alpha_n$. It is clear that T is transcendental and $\tilde{K} = \widetilde{Q_p(T)}$. It is easy to see that one has $0 < \omega(\varepsilon_n) \leqslant \frac{1}{p}$ (see Sect. 1, point 1). Inductively, one obtains $n - 1 < \omega(\alpha_n) < n$ for all $n \geqslant 2$. Let us define $d_i = \inf_x |T - x|$, where $x \in Q_p(\varepsilon_n)$. Observe that $d_i = \inf |\alpha_i - x|$, $x \in Q_p(\varepsilon_n)$ for n large enough. Let $i < n$. Then the element α_n has a conjugate α'_n with respect to the field $Q_p(\varepsilon_i)$, such that $v(\alpha_n - \alpha'_n) \leqslant i + 1$. Then by [5, 12, Corollary 2.12] it follows that $\inf_n |\alpha_n - x| > p^{-(i+1)}\rho$ (when x runs over $Q_p(\varepsilon_i)$), where ρ is a fixed real number. Then there results that the quotient $d_i / |Q_p(\varepsilon_i) : Q_p|$ does not tend to zero when $n \to \infty$. Hence T is not a Lipschitzian element.

3. A function $f\colon C(T)\to\mathbb{C}_p$ is called *Lipschitzian* if there exists a real number $c>0$ such that for any $x,\ y\in C(T)$ one has

$$|f(x)-f(y)|\leqslant c\,|x-y|.$$

THEOREM 4.4. *Let T be a Lipschitzian element of $\mathbb{C}_p$. Then any Lipschitzian function $f\colon C(T)\to\mathbb{C}_p$ is integrable with respect to the p-adic measure π_T.*

Proof. Let c be a real number such that $|f(x)-f(y)|\leqslant c\,|x-y|$ for all $x,\ y\in C(T)$. Let $\{\varepsilon_n\}_n$ be a strictly decreasing sequence of positive real numbers such that $\lim\varepsilon_n=0$ and that the sequence $\{\varepsilon_n/\varepsilon_{n+1}\}_n$ is bounded. For any $n\geqslant 1$ let $\{D(a_i^{(n)},\varepsilon)\}_n$ be an ε_n-covering of $C(T)$ and let us denote $A_n=\Phi(f,a_i^{(n)},\varepsilon_n)=\sum_{i=1}^{N(T,\varepsilon_n)} f(a_i^{(n)})\,\pi_T(D(a_i^{(n)},\varepsilon_n))=\sum_{i=1}^{N(T,\varepsilon_n)}(f(a_i^{(n)})/N(T,\varepsilon_n))$. Since $\varepsilon_n>\varepsilon_{n+1}$ it follows that for any i the ε_n-ball $D(a_i^{(n)},\varepsilon_n)$ is a disjoint union of, say g, ε_{n+1}-balls $\{D(a_{j(i)}^{(n+1)},\varepsilon_{n+1})\}_{j(i)\in J_i}$. Then we can write

$$\begin{aligned}A_n-A_{n+1}&=\sum_i\frac{f(a_i^{(n)})}{N(T,\varepsilon_n)}-\sum_j\frac{f(a_j^{(n+1)})}{N(T,\varepsilon_{n+1})}\\&=\left(\sum_i gf(a_i^{(n)})-\sum_{j\in J_i}f(a_j^{(n+1)})\right)\frac{1}{N(T,\varepsilon_{n+1})}.\end{aligned}$$

We have $|f(a_i^{(n)})-f(a_j^{(n+1)})|\leqslant c\,|a_i^{(n)}-a_j^{(n+1)}|<c\cdot\varepsilon_n$, and so $|A_n-A_{n+1}|<(c\cdot\varepsilon_n)/|N(T,\varepsilon_{n+1})|=(c\varepsilon_n/\varepsilon_{n+1})(\varepsilon_{n+1}/|N(T,\varepsilon_{n+1})|)\to 0$. Obviously for any $m\geqslant 1$ one has

$$\lim_n|A_n-A_{n+m}|=0.$$

Now let $\varepsilon>\varepsilon'>0$. There exist natural numbers m,n such that: $\varepsilon_{n-1}>\varepsilon\geqslant\varepsilon_n>\cdots>\varepsilon_{n+m-1}>\varepsilon'\geqslant\varepsilon_{n+m}$. Let $\{D(a_i^{(\varepsilon)}\varepsilon)\}$ and $\{D(a_i^{(\varepsilon')},\varepsilon')\}$ be the covering of $C(T)$ with ε-balls and ε'-balls, respectively. Also let $A_\varepsilon=\sum_{i=1}^{N(T,\varepsilon)}(f(a_i^{(\varepsilon)})/N(T,\varepsilon))$ and $A'_\varepsilon=\sum_{i=1}^{N(T,\varepsilon')}(f(a_i^{(\varepsilon')})/N(T,\varepsilon'))$ be the corresponding integral sums. Then as above one has $|A_\varepsilon-A_n|<(c\varepsilon_{n-1}/\varepsilon_n)(\varepsilon_n/|N(T,\varepsilon_n)|)\to 0$ and $|A_{n+m}-A_{\varepsilon'}|<(\varepsilon_{n+m}/|N(T,\varepsilon_{n+m})|)\to 0$ as $\varepsilon\to 0$. Hence one has $|A_\varepsilon-A_{\varepsilon'}|\to 0$ as $\varepsilon\to 0$. It follows that $\lim_{\varepsilon\to 0}A_\varepsilon$ is defined for any choice of the corresponding integral sums A_ε associated to the coverings of $C(T)$ with ε-balls. Finally, we obtain the existence of $\int_{C(T)}f\,d\pi_T$, as claimed.

We now turn to the question of the existence of Lipschitzian functions on $C(T)$. We say that a function $f\colon D\to\mathbb{C}_p$ where D is open, is a C^1 function if the derivative of f is defined and is continuous on D. One has the following result.

PROPOSITION 4.5. *Let D be an open set of $\mathbb{C}_p$ which contains $C(T)$. Then any C^1-function f: $D \to \mathbb{C}_p$ is Lipschitzian on $C(T)$.*

Proof. Since f', the derivative of f, is continuous on $C(T)$ it is bounded. Then for any $x \in C(T)$, there exists a ball $B[x, \varepsilon]$ included in D, such that for any $y \in B[x, \varepsilon]$ one has $|f(x)-f(y)|/|x-y| < |f'(x)| + \varepsilon \leqslant M_x$. Since $C(T)$ is compact, we can find finitely many such balls, say $\{B[x_i, \varepsilon_i]\}_i$ which cover $C(T)$. If x, y belong to distinct balls of the considered covering, then one has $|x-y| > \delta > 0$, and δ is independent of x and y. Denote $M = \sup_i\{M_{x_i}\}$. One has $|f(x)-f(y)| < M\,|x-y|$ if x, y belong to the same ball of the covering. If x, y do not belong to the same ball, then one has: $|x-y| > \delta > 0$, and so $|f(x)-f(y)| < M'\,|x-y|$ for a real number M' big enough. Let $c = \max\{M, M'\}$. Then $|f(x)-f(y)| < c\,|x-y|$ for any x, $y \in C(T)$.

COROLLARY 4.6. *Let T be a Lipschitzian element of $\mathbb{C}_p$. Any C^1-function f: $D \to \mathbb{C}_p$ where D is an open set which contains $C(T)$, is integrable on $C(T)$. In particular any polynomial function is integrable on $C(T)$.*

5. THE TRACE OF AN ELEMENT

In this section we give a definition for the trace of an element and investigate its existence.

1. Any element $\alpha \in \bar{Q}_p$ is a p.b.d.-element and so the p-adic measure π_α is bounded. Moreover for any function f: $C(\alpha) \to \mathbb{C}_p$ one has

$$\int_{C(\alpha)} f d_{\pi_\alpha} = \frac{1}{\deg(\alpha)} \sum f(\sigma(\alpha)).$$

Let us denote

$$Tr(\alpha) = \frac{1}{\deg(\alpha)}\, tr_{Q_p(\alpha)/Q_p}(\alpha) = \int_{C(\alpha)} x\, d\pi_\alpha.$$

We say that $Tr(\alpha)$ is the *trace* of α. More generally, if $T \in \mathbb{C}_p$ we shall denote

$$Tr(T) = \int_{C(T)} x\, d\pi_T$$

and call it the *trace* of T. We remark that this element does not always exist. If the last integral exists we shall say that T *has a trace* or that $Tr(T)$

exists. It is clear by Theorem 4.3 that any Lipschitzian element has a trace. Moreover, for $(*)$-elements one has the following result:

PROPOSITION 5.1. *Let T be a $(*)$-element of $\mathbb{C}_p$ and $\{\alpha_n\}_n$ be a $(*)$-sequence of $\bar{Q}_p$ such that $\lim_n \alpha_n = T$. Then $\{Tr(\alpha_n)\}$ is a convergent sequence of Q_p and one has $\lim_n Tr(\alpha_n) = Tr(T)$.*

Proof. Let $\varepsilon > 0$ be a real number and let n_0 be a natural number such that $|\alpha_{n+1} - \alpha_n|/\inf(|d_n|, |d_{n+1}|) < \varepsilon$ for $n \geqslant n_0$. Denote by $\mathscr{M}$ the set of all conjugates of α_n and α_{n+1}. Let us write the metric space $\mathscr{M}$ as a disjoint union of r balls of radius $|\alpha_{n+1} - \alpha_n|$. Then any ball contains the same number say "a" of conjugates of α_n and "b" conjugates of α_{n+1}. Here $d_n = ra$ and $d_{r+1} = rb$. Now for any $m \geqslant 1$ we have (n being the same as above)

$$Tr(\alpha_{n+1}) - Tr(\alpha_n) = \frac{1}{d_{n+1}} \sum_{\sigma} \sigma(\alpha_{n+1}) - \frac{1}{d_n} \sum_{\sigma} \sigma(\alpha_n).$$

Let D be the last common multiple of d_n and d_{n+1} and denote $A = D/d_n$, $B = D/d_{n+1}$. Then $Tr(\alpha_{n+1}) - Tr(\alpha_n) = \frac{1}{D}[B \sum_\sigma \sigma(\alpha_{n+1}) - A \sum_\sigma \sigma(\alpha_n)]$. Note that $B = Aa$, so for any of the above r balls we can put the "b" corresponding conjugates of α_{n+1} with multiplicities B in a on-to-one correspondence with the "a" conjugates of α_n with multiplicities A. Then $D(Tr(\alpha_{n+1}) - Tr(\alpha_n))$ will be equal to a sum of differences of the form $\sigma_1(\alpha_{n+1}) - \sigma_2(\alpha_n)$ with $|\sigma_1(\alpha_{n+1} - \sigma_2(\alpha_n))| \leqslant |\alpha_{n+1} - \alpha_n|$. Therefore $|D(Tr(\alpha_{n+1}) - Tr(\alpha_n))| \leqslant |\alpha_{n+1} - \alpha_n|$. Since $|D| = \inf(|d_n|, |d_{n+1}|)$, we get for $n \geqslant n_0$: $|Tr(\alpha_{n+1}) - Tr(\alpha_n)| \leqslant |\alpha_{n+1} - \alpha_n|/\inf(|d_n|, |d_{n+1}|) < \varepsilon$. Hence the sequence $\{Tr(\alpha_n)\}_n$ is Cauchy, so it converges in Q_p to $s(T)$, say. Now let $\varepsilon_n = |\alpha_{n+1} - \alpha_n|$ and let $f: \mathbb{C}(T) \to \mathbb{C}_p$ the inclusion function, $f(x) = x$, $x \in C(T)$. We may assume that the sequence $\{\varepsilon_n\}_n$ is strictly decreasing to zero. In the same way as above one can show that $|\Phi(\varepsilon_n, f) - Tr(\alpha_{n+1})| \leqslant \varepsilon_n$. Hence finally one gets

$$Tr(T) = \int_{C(T)} f\, d\pi_T = s(T) = \lim_n Tr(\alpha_n).$$

The proof is complete.

We will see that there are elements of $\mathbb{C}_p$ which are not Lipschitzian and still have a trace.

2. The following considerations will be useful later.

Let $T \in \mathbb{C}_p$. We shall use the notations of [2]. Denote $K_T = \widetilde{Q_p(T)} \cap \bar{Q}_p$, and $H(T) = \{\sigma \in G, \sigma(T) = T\}$.

PROPOSITION 5.2. *Let $T \in \mathbb{C}_p$. Then*:

(i) *$H(T)$ is a closed subgroup of G.*

(ii) *$H(T)$ is a normal subgroup if and only if K_T/Q_p is a normal extension.*

(iii) *If $T' \in \mathbb{C}_p$ then $H(T) \subseteq H(T')$ if and only if $\widetilde{Q_p(T)} \supseteq \widetilde{Q_p(T')}$.*

Proof. (i) Let $\sigma \in H(T)$. Then for any $f(T) \in Q_p(T)$ one has $\sigma(f(T)) = f(T)$ and since σ is continuous, one has $\sigma(z) = z$ for any $z \in \widetilde{Q_p[T]}$. Since $K_T = \bar{Q}_p \cap \widetilde{Q_p[T]}$ (see [2, Theorem 6.2]) then σ acts trivially on K_T. Conversely if σ acts trivially on K_T, then it acts trivially on $\tilde{K}_T = \widetilde{Q_p[T]}$ (see [2, Theorem 1.1]). Hence $\sigma \in H(T)$, and so $H(T) = H(K_T) = \{\sigma \in G, \sigma(x) = x$ for any $x \in K_T\}$. The result follows by infinite Galois Theory (see [4, p. 104]). By these considerations (ii) and (iii) also follow.

Let T, T' be elements of $\mathbb{C}_p$ such that $\widetilde{Q_p(T)} \supseteq \widetilde{Q_p(T')}$. Then according to Proposition 5.3, one has $H(T) \subseteq H(T')$ and then the map Ψ: $C(T) \to C(T')$ defined by $\Psi(\sigma(T)) = \sigma(T')$ is surjective and one has: $\Psi\varphi_T = \varphi_{T'}$ (see Remark 3.1). Since both $C(T)$ and $C(T')$ are endowed with the quotient topology, Ψ is continuous. According to [2, Theorem 6.3] one has $T' = \sum_{n \geqslant 0} a_n M_n(T)$, where $\{a_n\}_n$ is a sequence of p-adic numbers with limit zero and $\{M_n(T)\}_n$ is a suitable sequence of polynomials of $Q_p[T]$, ($\deg M_n(T) = n$). Hence $\Psi(\sigma(T)) = \sum_{n \geqslant 0} a_n M_n(\sigma(T)) = \sigma(T')$, for all $\sigma \in G$.

PROPOSITION 5.3. *Let K be an algebraic extension of Q_p, $K \subseteq \bar{Q}_p$. Assume that the trace function Tr: $K \to Q_p$ is continuous. Then any element $T \in \tilde{K}$ has a trace.*

The proof is obvious.

3. In general we do not know whether the function Tr is continuous on $K = K_T$. This happens if, for example, K is a p.b.d. extension. We shall describe another situation when the function Tr is continuous. If L/Q_p is a finite extension, let us denote by $\mathscr{D}_L$ the different of L/Q_p (see [4]).

PROPOSITION 5.4. *Let $Q_p \subset K \subseteq \bar{Q}_p$ be an extension. Assume that there exists a real number M such that $v(\mathscr{D}_L) - v([L : Q_p]) \geqslant M$ for any finite extension L/Q_p, $Q_p \subseteq L \subseteq K$. Then the function* Tr *is continuous on K.*

Proof. The function *Tr* is Q_p-linear. To show that Tr is continuous it is enough to prove its boundedness on the unit ball of K. Hence we must prove that there exists a real number M such that $v(Tr(x) - v(x)) \geqslant M$ for all $x \in O_K^*$ (the integers of K). Let $Q_p \subset L \subset K$ be such that L/Q_p is finite.

Let us put $tr = tr_{L/Q_p}$. For any $x \in L^*$, denote $e_x = [v(x)]$ (integral part) and let $M_L = \inf\{v(tr(x/p^{e_x})),\ x \in O_L^*\}$. According to the definition of $\mathcal{D}_L$, one knows (see [4]) that $\alpha \in \mathcal{D}_L^{-1}$ if and only if $tr(\alpha O_L) \subseteq Z_p$ (the p-adic integers). In other words, $v(\alpha) \geqslant -v(\mathcal{D}_L)$ if and only if $v(tr(\alpha O_L)) \geqslant 0$. Let us denote $m_L = [v(\mathcal{D}_L)]$. Then $v(\alpha) \geqslant -m_L$ implies $v(tr(\alpha)) \geqslant 0$ i.e. $v(\alpha p^{m_L}) \geqslant 0$ implies $v(tr(\alpha p^{m_L})) \geqslant m_L$. Since any $y \in O_L^*$ can be written as $y = \alpha p^{m_L}$, it follows that $m_L \leqslant \inf_y\{v(tr(y)),\ y \in O_L^*\}$. Therefore one necessarily has $M_L \geqslant m_L$. Since by hypothesis $v(\mathcal{D}_L) - v([L : Q_p]) \geqslant M$, for any finite extension $Q_p \subset L \subset K$, it follows $M_L - v([L : Q_p]) \geqslant M$. This implies the continuity of Tr on K.

Finally, we remark that if $T \in \mathbb{C}_p$ is such that for any $T' \in \widetilde{Q_p(T)}$, $T' = \sum_{n \geqslant 0} a_n M_n(T) = U(T)$, the function $U(z)$ is integrable on $C(T)$ with respect to p-adic measure π_T, then the function Tr is continuous on $K = K_T$.

An interesting situation when the function Tr is continuous is described in the following example.

Example 5.5. For any natural number n, let us denote by ε_n a primitive root of unity of degree p^n. Denote $K_n = Q_p(\varepsilon_n)$ and $K = \bigcup_{n \geqslant 1} K_n$. We remark that the set of rational numbers $\{v(\mathcal{D}_{K_n}) - v([K_n ; Q_p])\}$ is lower bounded. For that we observe that the polynomials $f = X^{p^{n-1}} - \varepsilon_1 \in Q_p(\varepsilon_1)[X]$ is irreducible over $Q_p(\varepsilon_1)$. This shows that $v(\mathcal{D}_{K_n/K_1}) = n - 1 = v([K_n : K_1])$. Then by Proposition 5.4 the function Tr is continuous on K.

Remark 5.6. (a) According to Example 4.2 there exist elements $T \in \tilde{K}$ which are not Lipschitzian. However, by Theorem 5.2, such elements T have a trace.

(b) There exist extensions K/Q_p, $Q_p \subset K \subseteq \bar{Q}_p$ such that the function Tr is not continuous on K. Indeed, for any natural number n, denote by $f_n = X^n + a_{n-1}X^{n-1} + a_{n-2}X^{n-1} + \cdots + a_n$ and Eisenstein polynomial such that $v(a_{n-1}) = 1$. Let α_n be a root of f_n for $n \geqslant 1$, and let $K = Q_p(\alpha_1, \alpha_2, \ldots \alpha_n, \ldots)$. Then Tr is not continuous on K if $v(n) \to \infty$ since it is not bounded on the unit ball.

6. THE FUNCTION $F(T, Z)$

1. Let T, z be elements of C_p such that $z \notin C(1/T)$. Then the function

$$f(x, z) \colon C(T) \to \mathbb{C}_p, \qquad x \mapsto \frac{1}{1 - xz}$$

is Lipschitzian. For, let us assume $z \neq 0$. Denote $d = \inf\{(1/z - x);\ x \in C(T)\}$. Then for any $x_1, x_2 \in C(T)$ one has

$$|f(x_1, z) - f(x_2, z)| = \frac{|z|\ |x_1 - x_2|}{|1 - zx_1|\ |1 - zx_2|} \leqslant \frac{|x_1 - x_2|}{|z|\ d^2},$$

and so $f(x, z)$ is Lipschitzian since $1/|z|\ d^2$ is a constant real number.

Now let us assume that T is a Lipschitzian element. Then for any $z \notin C(1/T)$ denote

$$F(T, z) = \int_{C(T)} f(x, z)\ d\pi_T = \int_{C(T)} \frac{1}{1 - xz}\ d\pi_T$$

(see Theorem 4.3). In this way we can define a function

$$F(T, Z)\colon \mathbb{C}_p \cup \{\infty\} \backslash C(1/T) \to \mathbb{C}_p$$

$$z \mapsto F(T, z), \qquad F(T, 0) = 1, \qquad F(t, \infty) = 0.$$

We call $F(T, Z)$ the *trace function associated to T*. Before going any further, let us mention that there are interesting connections between other work in p-adic integration and this paper. In particular, Barsky in [6] proves a general result which applies to the present paper in the case where T is a "p.b.d." element; in this case the series $F(T, Z)$ is essentially the Cauchy transform of the measure π_T. From this point of view the present work is an extension of Barsky's from p-adic measures to a type of "admissible" p-adic measure. We now prove the following:

THEOREM 6.1. *Let T be a Lipschitzian element of $\mathbb{C}_p$. Then $F(T, Z)$ is an equivariant rigid analytic function on $\mathbb{C}_p \cup \{\infty\} \backslash C(1/T)$. Any element of $C(1/T)$ is a singular point for $F(T, Z)$.*

Proof. For any $\delta > 0$ we denote $W_\delta = \mathbb{C}_p \cup \{\infty\} \backslash (C(1/T), \delta)$, where $B(C(1/T), \delta)$ is the open ball of radius δ around $C(1/T)$. Since $C(1/T)$ is compact, $B(C(1/T), \delta)$ will be a finite union of open balls and W_δ will be an affinoid contained in $\mathbb{C}_p \cup \{\infty\} \backslash C(1/T)$. To prove that $F(T, Z)$ is rigid analytic on $\mathbb{C}_p \cup \{\infty\} \backslash C(1/T)$ we need to show that it is rigid analytic on each W_δ. Fix a $\delta > 0$. We are done if we show that $F(T, Z)$ is the uniform limit on W_δ of a sequence of rational functions. The idea is to produce such rational functions by using the Riemannian sums associated with the integral which defines our function $F(T, Z)$. We proceed as follows.

Denote $f(z, x) = \frac{1}{1 - zx}$, $x \in C(T)$. Then one has

$$F(T, z) = \int_{C(T)} f(z, x)\ d\pi_T.$$

Since the last integral is well defined, it is a limit of Riemannian sums (see Remark 4.1(b)). Now let $\{\varepsilon_n\}_n$ be a strictly decreasing sequence of positive real numbers with limit zero and such that the sequence $\{\varepsilon_n/\varepsilon_{n+1}\}_n$ is upper bounded by a real number M. Let $\{D(a_i^{(n)}, \varepsilon_n)\}_n$ be an ε_n-covering of $C(T)$ and let us denote

$$A_n(z) = \Phi(f, a_i^{(n)}, \varepsilon_n) = \sum_i \frac{f(z, a_i^{(n)})}{N(T, \varepsilon_n)}.$$

We claim that the sequence $\{A_n(z)\}_n$ of rational functions of z converges uniformly to $F(T, z)$ on W_δ. Indeed, since $\varepsilon_n > \varepsilon_{n+1}$, it follows that for any i, the ε_n-ball $D(a_i^{(n)}, \varepsilon_n)$ is a disjoint union of, say g, ε_{n+1} balls $\{D(a_{j(i)}^{(n+1)}, \varepsilon_{n+1})\}_{j(i) \in J_i}$. Then we can write

$$\begin{aligned} A_n(z) - A_{n+1}(z) &= \sum_i \frac{f(z, a_i^{(n)})}{N(T, \varepsilon_n)} - \sum_j \frac{f(z, a_j^{(n+1)})}{N(T, \varepsilon_{n+1})} \\ &= \left(\sum_i \left(gf(z, a_i^{(n)}) - \sum_{j \in J_i} f(z, a_j^{(n+1)})\right)\right) \frac{1}{N(T, \varepsilon_{n+1})}. \end{aligned}$$

But one has

$$\begin{aligned} |f(z, a_i^{(n)}) - f(z, a_j^{(n+1)})| &= \left| \frac{1}{1 - za_i^{(n)}} - \frac{1}{1 - za_j^{(n+1)}} \right| \\ &= \left| \frac{z(a_i^{(n)} - a_j^{(n+1)})}{(1 - za_i^{(n)})(1 - za_j^{(n+1)})} \right| \\ &= |z| \, \frac{|a_i^{(n)} - a_j^{(n+1)}|}{|a_i^{(n)}| \, |a_j^{(n+1)}| \, |1/a_i^{(n)} - z| \, |1/a_j^{(n+1)} - z|} \\ &\leqslant \frac{|z| \, \varepsilon_n}{|T|^2 \, |1/a_i^{(n)} - z| \, |1/a_j^{(n+1)} - z|}. \end{aligned}$$

Note that if $|z| > |1/T|$ then $|1/a_i^{(n)} - z| = |1/a_j^{(n+1)} - z| = |z|$ and so $|z|/|1/a_i^{(n)} - z| \, |1/a_j^{(n+1)} - z| = 1/|z| < |T|$. If $|z| \leqslant |1/T|$ then we use the inequalities $|1/a_i^{(n)} - z| \geqslant \delta$ and $|1/a_j^{(n+1)} - z| \geqslant \delta$ to see that

$$\frac{|z|}{|1/a_i^{(n)} - z| \, |1/a_j^{(n+1)} - z|} \leqslant \frac{1}{|T| \, \delta^2}.$$

Therefore in both cases one has $|z|/|1/a_i^{(n)}-z|\,|1/a_j^{(n+1)}-z|\leqslant D$, where $D=\max\{|T|, 1/|T|\,\delta^2\}$. From this inequality we derive

$$|A_n(z)-A_n(z)|\leqslant\frac{D\varepsilon_n}{|T|^2}\cdot\frac{1}{|N(T,\varepsilon_{n+1})|}\leqslant\frac{DM}{|T|^2\,d^2}\;\frac{\varepsilon_{n+1}}{|N(T,\varepsilon_{n+1})|}.$$

Now, since T is a Lipschitzian element, one has $\varepsilon_{n+1}/|N(T,\varepsilon_{n+1})|\to 0$ and so $|A_n(z)-A_{n+1}(z)|\to 0$ uniformly with respect to $z\in W_\delta$. This shows that $F(T,Z)$ is a rigid analytic function on any W_δ and hence also on $\mathbb{C}_p\cup\{\infty\}\backslash C(1/T)$.

Finally, if $F(T,Z)$ has a rigid analytic continuation to a larger set, then since it is equivariant it would have a rigid analytic continuation to $\mathbb{C}_p\cup\{\infty\}$ and so by Liouville's Theorem it would be a constant. This is not possible since $F(T,\infty)=0$ while $F(T,0)=1$. The proof is complete.

As a consequence, note that on the open ball with radius $|1/T|$ centered in zero one has

$$F(T,Z)=\sum_{n\geqslant 0} Tr(T^n)\,Z^n.$$

2. Let $\alpha\in Q_p$ and let f be its (monic) minimal polynomial over Q_p. Then for any $z\in C_p-C(1/\alpha)$ one has: $F(\alpha,z)=\int_{C(\alpha)}(1/1-xz)\,d\pi_\alpha=\sum_\sigma 1/(1-\sigma(\alpha)\,z)=1/z\sum 1/(1/z-\sigma(\alpha))=f'(1/z)/dz\,f(1/z)$, $d=\deg(\alpha)$. Hence one has $F(\alpha,Z)=f'(1/Z)/dZ\,f(1/Z)=\sum_{n\geqslant 0} Tr(\alpha^n)\,Z^n$, where $Tr(\alpha^0)=1$. Moreover one has the following result:

Proposition 6.2. *Let $T\in\mathbb{C}_p$ be a $(*)$-element such that $|T|\leqslant 1$ and let $\{\alpha_n\}_n$ be a $(*)$-sequence such that $T=\lim\alpha_n$. Then for any $i\geqslant 0$ the sequence $\{Tr(\alpha_n^i)\}_n$ is convergent. Moreover the sequence $\{F(\alpha_n,Z)\}_n$ is uniformly convergent in the completion of $Q_p(Z)$ with respect to the Gauss valuation (see [1]) and for Z in a suitable ball centered in zero one has*

$$F(T,Z)=\lim_n F(\alpha_n,Z).$$

Proof. Clearly we may assume that $|T|=|\alpha_n|\leqslant 1$ for all $n\geqslant 1$. Further, as in the proof of Proposition 5.1 one shows that $Tr(T^s)=\lim_n Tr(\alpha_n^s)$ for all $s\geqslant 0$, from which one easily derives the stated result.

The following is a direct consequence of Theorem 6.1.

Proposition 6.3. *Let T be a Lipschitzian element. Then an element $U\in\mathbb{C}_p$ belongs to $C(T)$ (i.e., is conjugate with T) if and only if U is also a Lipschitzian element and for Z in a ball centered in zero one has*

$$F(T,Z)=F(U,Z).$$

3. We end this section with a few comments. Let K be an algebraic extension of Q_p, $K \subseteq \bar{Q}_p$, such that the trace function Tr: $K \to Q_p$ is continuous. Then for any element $T \in \tilde{K}$ one can define $Tr(T)$ and so one can define the function $F(T, Z)$ which verifies Theorem 6.1. The details are left to the reader.

Although the trace function $F(T, Z)$ can be defined for a wider set of elements T of $\mathbb{C}_p$, the methods employed in the present paper work nicely mainly for $(*)$-elements (see next sections). This study will be extended in a forthcoming paper which relies more heavily on integration theory on sets like $C(T)$, $T \in \mathbb{C}_p$.

7. THE BEHAVIOR OF $F(T, Z)$ AROUND $C(1/T)$

1. By Theorem 6.1 it follows that $F(T, Z)$ belongs to $A(\mathbb{C}_p \cup \{\infty\} \backslash C(1/T))$. As a direct consequence one has

COROLLARY 7.1. *Let T be a Lipschitzian transcendental element of $\mathbb{C}_p$ over Q_p. Then $F(T, Z)$ is not a rational function (i.e., it does not belong to $\mathbb{C}_p(Z)$).*

The result follows from the fact that $C(1/T)$, the set of singularities of $F(T, Z)$ is not a finite set (see Proposition 5.3). Our goal is to prove the following:

THEOREM 7.2. *Let $\{T_\alpha\}_\alpha$ be a family of $(*)$-elements of $\mathbb{C}_p$ which are transcendental over Q_p and any two are non-conjugate. Then the functions $\{F(T_\alpha, Z)\}_\alpha$ are algebraically independent over $\mathbb{C}_p(Z)$.*

As a consequence one has the following:

COROLLARY 7.3. *Let T be a $(*)$-element of $\mathbb{C}_p$. Then $F(T, Z)$ is transcendental over $\mathbb{C}_p(Z)$ if and only if T is transcendental over Q_p.*

We start by introducing the following condition:

(E) Let $T \in \mathbb{C}_p$ be a $(*)$-element. Then for any $S \in C(1/T)$ there exists a sequence $\{\beta_n\}_n \subseteq C_p \backslash C(1/T)$ so that $\lim_n \beta_n = S$, and that $\lim_n |F(T, \beta_n)| = \infty$.

Let us assume that condition (E) holds and prove Theorem 7.2. We need to show that for any non-zero polynomial P in r variables over $\mathbb{C}_p$ and any distinct elements $T_{\alpha_1}, \ldots, T_{\alpha_{r-1}}$ in the given family the function defined by $Z \mapsto P(Z, F(T_{\alpha_1}, Z), \ldots, F(T_{\alpha_{r-1}}, Z))$ does not vanish identically. We prove this by induction on r. The statement is clear for $r = 1$. Let us take an $r > 1$

and assume the statement for 1, 2, ..., r. Suppose there exist distinct elements $T_{\alpha_1}, ..., T_{\alpha_r}$ and a non-zero polynomial P in $r+1$ variables over $\mathbb{C}_p$ such that

$$P(Z, F(T_{\alpha_1}, Z), ..., F(T_{\alpha_r}, Z)) = 0.$$

We write this equality in the form

$$A_0 F^m + A_1 F^{m-1} + \cdots + A_m = 0,$$

where $A_0, ..., A_m$ are polynomials in Z, $F(T_{\alpha_1}, Z), ..., F(T_{\alpha_{r-1}}, Z)$ with coefficients in $\mathbb{C}_p$, $F = F(T_{\alpha_r}, Z)$ and A_0 is not the zero polynomial. Then by the induction hypothesis the function

$$Z \mapsto A_0(Z, F(T_{\alpha_1}, Z), ..., F(T_{\alpha_{r-1}}, Z))$$

does not vanish identically. Since this is a rigid analytic function defined in a neighborhood of $C(1/T_{\alpha_r})$ (its set of singular points being contained in the union of orbits $C(1/T_{\alpha_1}), ..., C(1/T_{\alpha_{r-1}})$) only finitely many elements of $C(1/T_{\alpha_r})$ will be zeros of it. We choose an element $S \in C(1/T_{\alpha_r})$ such that A_0 does not vanish at S and then we let Z approach S. The functions

$$Z \mapsto A_j(Z, F(T_{\alpha_1}, Z), ..., F(T_{\alpha_{r-1}}, Z)), \qquad 0 \leqslant j \leqslant m$$

are rigid analytic in a neighborhood of S and by condition (E) one can find a sequence $Z_n = \beta_n$ convergent to S such that $|F(T_{\alpha_r}, Z_n)| \to \infty$. Then since $A_0(S, F(T_{\alpha_1}, S), ..., F(T_{\alpha_{r-1}}, S)) \neq 0$ we see that as $n \to \infty$ the term $A_0 F^m$ with $Z = Z_n$ dominates in absolute value the sum $A_1 F^{m-1} + \cdots + A_m$. This contradicts our assumption that $P(Z, F(T_{\alpha_1}, Z), ..., F(T_{\alpha_r}, Z))$ vanishes identically, which completes the proof of Theorem 7.2.

Remark 7.4. In Theorem 7.2 one may replace each of the functions $\{F(T_\alpha, Z)\}$ by one of its derivatives (with respect to Z) $F^{(i_\alpha)}(T_\alpha, Z)\}$. Indeed, as we shall see later condition (E) holds also for the derivatives $F^{(i)}(T, Z)$.

2. We end this section with some remarks and comments. Here $T = \lim_n \alpha_n$, where $\{\alpha_n\}_n$ si a $(*)$-sequence of $\bar{Q}_p$ and f_n is the monic minimal polynomial of α_n over Q_p.

(a) Let us denote by $\mathscr{L}$ the set of all polynomials $f'_n(1/X)$, $n \geqslant 1$, and denote by $\mathscr{L}'$ the set of all limit points of $\mathscr{L}$. Let $z \in \mathscr{L}'$ and let us assume that $z \notin C(1/T)$. Since $f_n(1/z) \neq 0$ for n big enough, we obtain: $F(T, z) = 0$. Let ε be such that $B(z, \varepsilon) \cap C(1/T) = \varnothing$, and let $\alpha \in B(z, \varepsilon) \cap \bar{Q}_p$. Then we can write $F(T, Z) = \sum_{n \geqslant 0} c_n (Z - \alpha)^n$, $a_n \in Q_p(\alpha)$, $n \geqslant 0$. Since $\sum_n c_m (z - \alpha)^n = 0$, it follows that $z \in \bar{Q}_p$ and so the set of elements of $\mathscr{L}' \cap B(\alpha, \varepsilon)$ is necessarily finite and all belong to $\bar{Q}_p$. Moreover, for any ball $B[x, \varepsilon]$ such that $B(x, \varepsilon) \cap C(1/T) = \phi$ there exist only a finite number of elements of $\mathscr{L}'$ in $B[x, \varepsilon]$.

(b) We now give an example of a transcendental element T such that $F(T, Z)$ has zeros in every neighborhood of any of its singular points. This example together with condition (E) shows that any singular point of $F(T, Z)$ is in fact an "essential" singular point.

Denote $p_1 = 2, p_2, ...p_n, ...$ the (increasing) sequence of all prime numbers. For any $n \geqslant 1$ we shall construct a polynomial f_n of degree $p_1 \cdots p_n = q_n$, and a root α_n of it such that:

(i) The polynomial f_n is irreducible and the extension $Q_p(\alpha_n)/Q_p$ is fully ramified.

(ii) If $\omega(\alpha_n) = \sup(v(\alpha_n - \alpha'_n))$, where α'_n runs over all the conjugates of α_n, $\alpha'_n \neq \alpha_n$, then $v(\alpha_{n+1} - \alpha_n) \geqslant \omega(\alpha_n) + n$.

Let us take $f_1 = X^2 - p$ and let α_1 be a root of f_1. Assume that $n \geqslant 1$, and that $f_1, ..., f_n, \alpha_1, ..., \alpha_n$ have already been defined such that the conditions (i) and (ii) are verified. Then we shall define f_{n+1} and α_{n+1}.

Let s_n be the integral part of $\omega(\alpha_n)$ and let $f_{n+1} = f_n^{p_{n+1}} - p^{q_{n+1}(s_n+n)+1}$. One has $\deg f_{n+1} = q_{n+1}$. Since $v(f_{n+1}(\alpha_n)) = q_{n+1}(s_n + n) + 1$, there exists at least a root α_{n+1} of f_{n+1} such that $v(\alpha_{n+1} - \alpha_n) \geqslant s_n + n > \omega(\alpha_n)$. Then according to Krasner's Lemma (see [4, p. 44]), one has $Q_p(\alpha_n) \subseteq Q_p(\alpha_{n+1})$. Furthermore, one has $v(f_n(\alpha_{n+1})) = q_n(s_n + n) + 1/q_{n+1}$. By this equality and the inductive hypothesis it follows

$$e(Q_p(\alpha_{n+1})/Q_p) = q_{n+1}.$$

This show that f_{n+1} is irreducible. It is clear that the sequence $\{\alpha_n\}$ just defined is Cauchy and verifies condition $(*)$ stated in Section 4. If $T = \lim \alpha_n$ then the trace function $F(T, Z)$ is defined and one has $F(T, Z) = \lim_n F(\alpha_n, Z)$ (see Proposition 6.2). Now for any $n \geqslant 1$, one has $F(\alpha_{n+1}, Z) = f'_{n+1}(1/Z)/q_{n+1} Z f_{n+1}(1/Z)$. Then according to the inductive definition of f_{n+i} we have $f'_{n+i}(1/\alpha_n) = 0$ and so $F(\alpha_{n+i}, 1/\alpha_n) = 0$, for all $i \geqslant 1$. Hence $F(T, 1/\alpha_n) = 0$ for all $n \geqslant 1$. Since the sequence α_n converges to T, the sequence $1/\alpha_n$ converges to $1/T$. The result follows from Theorem 6.1.

8. CONTINUITY PROPERTIES OF μ_T

1. Let f: $\mathbb{C}_p \to \mathbb{C}$ be a function where $\mathbb{C}$ is the field of complex numbers. Let us assume that f is integrable with respect to several measures μ_T. We then ask whether there is any relationship between the integrals

$\int_{\mathbb{C}_p} f\,d\mu_T$ for various elements $T \in \mathbb{C}_p$. More precisely, let $\{\alpha_n\}_n$ be a sequence of elements of $\mathbb{C}_p$ and let $T = \lim_n \alpha_n$. Is it true that the sequence $\{\mu_{\alpha_n}\}_n$ is weakly convergent to μ_T?

This question has an affirmative answer in the following interesting context. Let $\{\alpha_n\}_n$ be a sequence of elements of $\bar{Q}_p$ and let $T = \lim_n \alpha_n$. Let $\varepsilon > 0$ be a real number and let us consider the set $B(C(T), \varepsilon)$ (see Section 2). Denote by $\mathscr{C}(B(C(T), \varepsilon))$ the set of all functions defined on $B(C(T), \varepsilon)$ with complex values, and which are continuous and bounded. For any $f \in \mathscr{C}(B(C(T), \varepsilon))$, denote $\|f\| = \sup_{z \in B(C(T), \varepsilon)} |f(z)|$. In this way $\mathscr{C}(B(C(T), \varepsilon))$ becomes a Banach algebra over the field $\mathbb{C}$ of complex numbers. Since $\alpha_n \to T$, for n big enough one has: $C(\alpha_n) \subseteq B(C(T), \varepsilon)$. For such n let us denote φ_n: $\mathscr{C}(B(C(T), \varepsilon)) \to \mathbb{C}$ the mapping defined by

$$\varphi_n(f) = \int_{B(C(T), \varepsilon)} f\,d\mu_{\alpha_n}.$$

Then φ_n is linear and continuous and one has

$$|\varphi_n(f)| \leqslant \int_{B(C(T), \varepsilon)} |f|\,d\mu_{\alpha_n} \leqslant \|f\|\,\mu_{\alpha_n}(B(C(T), \varepsilon)) = \|f\|.$$

Since for some f one has equality (for example, if f is constant), it follows that $\|\varphi_n\| = \sup_f (|\varphi_n(f)|/\|f\|) = 1$. On the other hand, we may also consider the functional φ: $\mathscr{C}(B(C(T), \varepsilon)) \to \mathbb{C}$ defined by

$$\varphi(f) = \int_{B(C(T), \varepsilon)} f\,d\mu_T.$$

We ask whether the sequence $\{\varphi_n\}_n$ is convergent to φ in the so-called "weak convergence" (see [13, p. 233]). According to Alaoglu's Theorem (see [13]) one knows that the unit ball $\mathscr{U}$ of the dual of $\mathscr{C}(B(C(T), \varepsilon))$ is compact in the weak topology. Hence the sequence $\{\varphi_n\}_n$ (which is included in $\mathscr{U}$ for n big enough) will have a subsequence which (weakly) converges to an element ψ of $\mathscr{U}$. We will show that $\psi = \varphi$. Since both φ and ψ are continuous it will be enough to show that $\varphi(f) = \psi(f)$ for f in a dense subset of $\mathscr{C}(B(C(T), \varepsilon))$. This set will be denoted by $\mathscr{E}$ and will be defined as follows: An element $f \in \mathscr{C}(B(C(T), \varepsilon))$ belongs to $\mathscr{E}$ if there exists a neighborhood V of $C(T)$ such that the restriction of f to V is locally constant, i.e., has only a finite number of values.

Let $f \in \mathscr{E}$. Then there exist finitely many balls $\{B(a_i, \varepsilon_i)\}_i$, included in $B(C(T), \varepsilon)$, such that $C(T) \subseteq \bigcup_i B(a_i, \varepsilon_i)$ and that f is constant on $B(a_i, \varepsilon_i)$ for all i. Then for n big enough one has $|\alpha_n - T| < \varepsilon_i$ for all i, and so according to Proposition 3.6 one has $\mu_T(B(a_i, \varepsilon_i)) = \mu_{\alpha_n}(B(a_i, \varepsilon_i))$ for all i. This means that the sequence $\{\varphi_n(f)\}_n$ is ultimately constant and $\varphi_n(f)=\varphi(f)$ for n big enough. Hence $\varphi(f)=\psi(f)$ for all $f\in\mathscr{E}$. To show the equality $\varphi=\psi$ it is enough to prove that $\mathscr{E}$ is dense in $\mathscr{C}(B(C(T),\varepsilon))$. For, let $f\in\mathscr{C}(B(C(T),\varepsilon))$ and let $\varepsilon_1>0$ be a real number. For any $z\in C(T)$ let us take an open ball V_z centered in z and included in $f^{-1}(B(f(z),\varepsilon_1))$. Let $\{z_i\}_{1\leqslant i\leqslant k}$ be such that $\bigcup_i V_{z_i}\supseteq C(T)$ and that $V_{z_i}\cap V_{z_j}=\varnothing$ if $i\neq j$. Denote by f_{ε_1}: $B(C(T),\varepsilon)\to C$ the function defined by: $f_{\varepsilon_1}(x)=f(z_i)$ if $x\in V_{z_i}$ and $f_{\varepsilon_1}(x)=f(x)$ if $x\neq\bigcup_i V_{z_i}$. It is clear that f_{ε_1} is continuous and that $\|f-f_{\varepsilon_1}\|<\varepsilon_1$. Since ε_1 is arbitrary, it follows that $\mathscr{E}$ is dense in $\mathscr{C}(B(C(T),\varepsilon))$. This answers the question raised at the beginning of this section.

THEOREM 8.1. *Let $\{\alpha_n\}_n$ be a convergent sequence of elements of $\bar{Q}_p$ and let $T=\lim_n\alpha_n$. Then for any $\varepsilon>0$ the sequence $\{\mu_{\alpha_n}\}$ is weakly convergent to μ_T on the Banach algebra $\mathscr{C}(B(C(T),\varepsilon))$.*

2. We end this section with some remarks. If $0<\varepsilon_1<\varepsilon_2$ are real numbers, then one has $B(C(T),\varepsilon_1)\subset B(C(T),\varepsilon_2)$ and so by restriction one obtains a homomorphism of algebras,

$$\rho_{\varepsilon_1,\varepsilon_2}\colon\mathscr{C}(B(C(T),\varepsilon_2))\to\mathscr{C}(B(C(T),\varepsilon_1)).$$

If $\varepsilon\to 0$, we can consider the inductive limit of algebras,

$$O_{\text{cont}}(C(T))=\lim_{\vec{\varepsilon}}\mathscr{C}(B(C(T),\varepsilon))$$

(the germs algebra around $C(T)$).

Let $\{\alpha_n\}_n$ be a sequence of $\bar{Q}_p$ whose limit is T and let $f\in O_{\text{cont}}(C(T))$. Let $\varepsilon_0>0$, and let $F\in\mathscr{C}(B(C(T),\varepsilon_0))$ be such that the image of F in the inductive limit is just f. According to Theorem 8.1, for any $\varepsilon\geqslant\varepsilon_0$ the sequence $\{\int_{B(C(T),\varepsilon)}F\,d\mu_{\alpha_n}\}_n$ is convergent to $\int_{B(C(T),\varepsilon)}F\,d\mu_T$, and moreover one has the equality $\int_{B(C(T),\varepsilon)}F\,d\mu_{\alpha_n}=\int_{B(C(T),\varepsilon)}F\,d\mu_T$ for n big enough. Let us define

$$\int_{C(T)}f\,d\mu_{\alpha_n}=\lim_{\varepsilon\leqslant\varepsilon_0}\int_{B(C(T),\varepsilon)}F\,d\mu_{\alpha_n},$$

$$\int_{C(T)}f\,d\mu_T=\lim_{\varepsilon\leqslant\varepsilon_0}\int_{B(C(T),\varepsilon)}F\,d\mu_T.$$

It is easy to see that $\int_{C(T)}f\,d\mu_{\alpha_n}$ and $\int_{C(T)}f\,d\mu_T$ do not depend on the choice of ε_0 and F, and one has

$$\int_{C(T)}f\,d\mu_T=\lim_n\int_{C(T)}f\,d\mu_{\alpha_n}. \tag{1}$$

9. THE METRIC INVARIANT $\Delta(T)$

1. Let $T \in \mathbb{C}_p$. Denote $f = f_T \colon \mathbb{C}_p \to \mathbb{R}$ the function defined by $f(z) = v(z - T)$. The function f is not defined in T and is a locally constant function. Precisely, for any $z_0 \in \mathbb{C}_p$, $z_0 \neq T$, f is constant on the ball $B(z_0, \varepsilon(f(z_0)))$. For any real number δ we shall denote

$$\varepsilon(\delta) = p^{-\delta}. \tag{2}$$

It is clear that f is integrable with respect to the measure μ_T, and let us denote

$$\Delta(T) = \int_{\mathbb{C}_p} f \, d\mu_T = \int_{\mathbb{C}_p} v(z - T) \, d\mu_T.$$

Since one has $f_{\sigma(T)}(z) = f(\sigma^{-1}(z))$ it follows that $\Delta(T)$ does not depend on σ, and so it is an invariant with respect to G.

We shall say that T is of the *first kind* if $\Delta(T)$ is a finite number. Otherwise we shall say that T is of the *second kind*. Any element of $\bar{Q}_p$ is of the second kind. The number T defined in Example 3.5 is of the first kind. Indeed, for any $n \geqslant 0$ it is easy to see that $\int_{\mathbb{C}_p} v(z - T) \, d\mu_{\alpha_n} = 2 - 1/2^n$, and so according to Theorem 8.1 one has $\int_{\mathbb{C}_p} v(z - T) \, d\mu_T = 2$.

Remark 9.1. Let $Q_p \subset K \subseteq \bar{Q}_p$ be such that the extension K/Q_p is infinite. There exists an element $T \in \mathbb{C}_p$ such that T verifies condition $(*)$, $\widetilde{Q_p(T)} = \tilde{K}$ and $\Delta(T) = \infty$.

For, let $\{\alpha_n\}_n$ be a sequence of elements of K such that $Q_p(\alpha_n) \subset Q_p(\alpha_{n+1})$ for all $n \geqslant 1$ and $\bigcup_n Q_p(\alpha_n) = K$. We shall define a sequence $\{a_n\}_n$ of elements of Q_p and a sequence $\{\beta_n\}_n$ of elements of K as follows. We set

$$\alpha_1 = \beta_1, \qquad a_1 = 1.$$

Let $n \geqslant 1$ and assume that the elements $a_1, ..., a_n$ and $\beta_1, ..., \beta_n$ have been defined. We shall put $\beta_{n+1} = \beta_n + a_{n+1}\alpha_{n+1}$.

Let us choose a_{n+1} such that:

(1) $Q_p(\alpha_{n+1}) = Q_p(\beta_{n+1})$;

(2) if $n \geqslant 2$, $v(\beta_{n+1} - \beta_n) > \omega(\beta_n)$ and $v(\beta_{n+1} - \beta_n) > v(\beta_n - \beta_{n-1})$ (here $\omega(\beta_n) = \sup v(\beta_n - \beta'_n)$, where β'_n runs over all the conjugates of β_n distinct to β_n);

(3) $v(\beta_{n+1} - \beta_n)/d_{n+1} \to \infty$, $d_n = \deg \alpha_n$.

Then it is easy to see that the sequence $\{\beta_n\}$ is convergent. Let $T = \lim_n \beta_n$. By conditions (2) and (3) it follows that $\widetilde{Q_p(T)} = \tilde{K}$ (see [11]) and that T verifies condition $(*)$. Moreover one has $v(T-\beta_n)-v(d_n)\to\infty$.

Let us denote $\delta_n = v(T-\beta_n)$ and let $\varepsilon_n = p^{-\delta_n}$. Then the number $N(T,\varepsilon_n)$ is a divisor of d_n (see Remark 3.8). If $\varepsilon_n\to 0$ then $v(T-\beta_n)\,\mu_T(B(\beta_n,\varepsilon_n))\}_n$ tends to $\Delta(T)$. This shows that $\Delta(T)=\infty$, as claimed.

By this remark it follows that it is possible to construct two transcendental numbers T, T' of $\mathbb{C}_p$ which verify the condition $(*)$ such that $\widetilde{Q_p(T)} = \widetilde{Q_p(T')}$, T is of the first kind and T' is of the second kind. This shows that the number $\Delta(T)$ defined above is not a topological invariant. It is however a metric invariant in the usual sense.

We now proceed to study the invariant $\Delta(T)$. If δ is a real number, let us denote by $f_\delta\colon \mathbb{C}_p\to\mathbb{R}$ the function defined by $f_\delta(z)=\inf(v(z-T),\delta)$. If $\{\delta_n\}_n$ is a sequence of real numbers which tends to infinity, the sequence $\{f_{\delta_n}(z)\}$ will be convergent to $f(z)$. Then, according to Lebesque's theorem of monotone convergence (see [13]), it follows that the sequence $\{\int_{\mathbb{C}_p} f_{\delta_n}\,d\mu_T\}_n$ is convergent to $\int_{\mathbb{C}_p} f\,d\mu_T$.

Let us remark that for any δ the function f_δ is constant on the ball $B[z,\varepsilon(\delta)]$. Hence, if $T_1\in B(T,\varepsilon(\delta))$ then by Proposition 3.6 and the fact that $\inf(v(z-T_1),\delta)=\inf(v(z-T),\delta)$ it follows

$$\Delta_\delta(T)=\int_{\mathbb{C}_p}\inf(v(z-T),\delta)\,d\mu_T=\int_{\mathbb{C}_p}\inf(v(z-T_1),\delta)\,d\mu_T.$$

Let $\{T_n\}_n$ be a sequence in $\mathbb{C}_p$ such that $\lim T_n=T$. By the last equality we get that for any δ and large enough n, one has $\Delta_\delta(T)=\int_{\mathbb{C}_p}\inf(v(z-T_n),\delta)\,d\mu_{T_n}$, and so $\Delta_\delta(T)\leqslant \lim_n\inf\Delta_\delta(T_n)$. Since this is true for all δ, it follows that $\Delta(T)\leqslant\lim_n\inf\Delta(T_n)$. This shows that the map $\Delta\colon\mathbb{C}_p\to R\cup\{\infty\}$ is uniformly semicontinuous. Since $\Delta(\alpha)=\infty$ for any $\alpha\in\bar{Q}_p$, it follows that the map Δ is not continuous, since there exist elements T of the first kind!

2. Let $\alpha\in\bar{Q}_p$ be such that $v(\alpha-T)>\delta$. Denote by h the monic minimal polynomial of α over Q_p. Denote by w_δ the extension of v to $\mathbb{C}_p(X)$, (X an indeterminate) defined by the pair (T,δ) (see [2]). Then one has: $w_\delta(h)=\sum_{\sigma(\alpha)}\inf(\delta, v(T-\sigma(\alpha)))$, where $\sigma(\alpha)$ runs over all the conjugates of α. On the other hand, according to Proposition 3.6, one has $\Delta_\delta(T)=\int_{\mathbb{C}_p}\inf(v(z-T),\delta)\,d\mu_T=\int_{\mathbb{C}_p}\inf(v(z-T),\delta)\,d\mu_\alpha=\sum_{C(\alpha)}\inf(v(T-\sigma(\alpha)),\ \delta)(1/\deg h)=(w_\delta(h)/\deg h)$. Hence $w_\delta(h)=\deg(h)\,\Delta_\delta(T)$.

Let $h'(X)=\prod_j P_j(X)$ be the decomposition of $h'(X)$ in the ring $Q_p[X]$. For any j, denote by β_j the root of P_j which is nearest to T. If $v(T-\beta_j)\geqslant\delta$, then as above one has $w_j(P_j)=(\deg P_j)\,\Delta_j(T)$. If $v(\beta_j-T)<\delta$, then we claim that one has

$$w_\delta(P_j)\leqslant\deg(P_j)\,\Delta_\delta(T). \tag{3}$$

In order to prove the claim, let $\lambda_j = v(T-\beta_j)$ and denote by w_j the extension of v to $\mathbb{C}_p(X)$ defined by the pair (T, λ_j) (see [2]). Then one has: $w_j(P_j) = \sum_{\sigma(\beta_j) \in C(\beta_j)} \inf(v(T-\sigma(\beta_j)), \lambda_j)$. We get $w_\delta(T-\sigma(\beta_j)) = \inf(v(T-\sigma(\beta_j)), \delta) = v(T-\sigma(\beta_j)) = \inf(v(T-\sigma(\beta_j)), \lambda_j) = w_j(T-\sigma(\beta_j))$, and so $w_\delta(P_j) = w_j(P_j)$. We derive: $w_\delta(P_j) = w_j(P_j) = \sum_{\sigma(\beta_j) \in C(\beta_j)} \inf(v(T-\sigma(\beta_j)), \lambda_j) = \deg(P_j) \int_{\mathbb{C}_p} \inf(v(T-z), \lambda_j)\, d\mu_{\beta_j} = \deg(P_j) \int_{\mathbb{C}_p} \inf(v(T-z), \lambda_j))\, d\mu_T = \deg(P_j)\, \Delta_{\lambda_j}(T)$. Since $\lambda_j < \delta$, we have $\Delta_{\lambda_j}(T) \leqslant \Delta_j(T)$, and this completes the proof of the claim.

Furthermore, for $h'(X)$ one has

$$w_\delta(h'(X)) = \sum_{jw_j} (P_j(X)) \leqslant \sum_j \deg\,(P_j)\, \Delta_\delta(T) = \deg(h')\, \Delta_\delta(T).$$

Hence $w_\delta(h/h') \geqslant \deg(h)\, \Delta_\delta(T) - \deg(h')\, \Delta_\delta(T) = \Delta_j(T)$.

Let $T \in \mathbb{C}_p$ be a transcendental element over Q_p. Assume that $\{\alpha_n\}$ is a sequence of elements of $\bar{Q}_p$ such that $T = \lim_n \alpha_n$, and that it satisfies the condition $(*)$. Assume that T is of the second kind. For any n we set $\delta_n = v(T-\alpha_n)$. Denote by w_n the extension of v to $\mathbb{C}_p(X)$ defined by the pair (T, δ_n). It is clear that we can assume $\delta_n \leqslant \delta_{n+1}$ for all n. If P is any polynomial of $\mathbb{C}_p[X]$, one has $w_n(P) = \sum_\beta \inf(v(T-\beta), \delta_n)$, where β runs over all the roots of P, and so $w_n(P) \leqslant w_{n+1}(P)$. Moreover $v(P(T)) = \sup_n w_n(P)$, for all $P \in \mathbb{C}_p[X]$.

Let f_n be the minimal polynomial of α_n. By the above considerations for any n one gets

$$v(f_n(T)/f'_n(T)) \geqslant w_n(f_n/f'_n) \geqslant \Delta_{\delta_n}(T)$$

and so

$$\sup_n v(f_n(T)/f'_n(T)) \geqslant \sup_n \Delta_{\delta_n}(T) \geqslant \Delta(T) = \infty.$$

From this inequality it follows that

$$\inf_n \left| \frac{f'_n(T)}{f_n(T)} \right| = \infty$$

and so, if we denote $d_n = \deg\,(f_n)$, we have

$$\inf_n \left| \frac{f'_n(T)}{d_n(1/T)\, f_n(T)} \right| = \infty.$$

We know (see Proposition 6.2) that $F(T, Z) = \lim_n (f'_n(1/Z)/d_n Z\, f_n(1/Z))$. Also, it is easy to see that $\{1/\alpha_n\}_n$ verifies condition $(*)$ and the element $1/T = \lim\,(1/\alpha_n)$ is of the second kind. Then, as above one has

$$\inf_n \left| \frac{f'_n(1/T)}{d_n T\, f_n(1/T)} \right| = \infty.$$

This shows that if Z approaches $1/T$ then $F(T, Z)$ takes values as large as we want. By these considerations it follows that condition (E) holds for any T of the second kind. Hence Theorem 7.2 is proved for T of the second kind.

10. PROOF OF THEOREM 7.2

Let $\{\alpha_n\}_n$ be a sequence of elements of $\bar{Q}_p$ which verifies the condition $(*)$ (see Section 4) and let $T = \lim_n \alpha_n$. Assume T is transcendental over Q_p. We already know that if T is of the second kind then the condition (E) is verified and so Theorem 7.2 is valid for such T. We now assume that T is of the first kind.

Let f_n be the minimal polynomial of α_n and let $f'_n = \prod_j P_j$ be the irreducible factor decomposition of $f'_n(X)$ in $Q_p[X]$. As usually denote $d_n = \deg\,(f_n)$. Also for any j let us denote $\lambda_j = v(T - \beta_j) = \max(v(T - \beta))$, where β runs over all the roots of P_j. Let δ be a real number and let n be such that $v(T - \alpha_n) \geqslant \delta$. As in the previous section, denote $w = w_\delta$ (respectively w_j) the extension of v to $\mathbb{C}_p(X)$ defined by the pair (T, δ) (respectively (T, λ_j)). By the results of the previous section one gets

$$\begin{aligned} w\left(\frac{f_n(X)}{f'_n(X)}\right) &= d_n\, \Delta_\delta(T) - \sum_j \deg\,(P_1)\, \Delta_{\lambda_j}(T) \\ &= \Delta_\delta(T) + \sum_j \deg\,(P_j)(\Delta_\delta(T) - \Delta_{\lambda_j}(T)). \end{aligned} \tag{4}$$

It is clear that the right hand side of (4) is a sum of positive terms. Our aim is to show that this sum tends to infinity when δ tends to infinity. Since by the hypothesis that T is of the first kind one has $\Delta_\delta(T) \leqslant \Delta(T) < \infty$, this possibility is only given by the sum $\sum_j \deg\,(P_j)\,(\Delta_\delta(T) - \Delta_{\lambda_j}(T))$. Also, since $\Delta_{\lambda_j}(T) = \Delta_\delta(T)$ if $\lambda_j \geqslant \delta$, we will focus on the indices j such that $\lambda_j < \delta$. Let j be such that $\lambda_j < \delta$. Then one has

$$\Delta_\delta(T) - \Delta_{\lambda_j}(T) = \int_{\mathbb{C}_p} (\inf(v(z-T), \delta) - \inf(v(z-T), \lambda_j))\, d\mu_T. \tag{5}$$

Let $\varepsilon = \varepsilon(\delta)$, $\varepsilon_j = \varepsilon(\lambda_j)$ (see (2)). Then $B[T, \varepsilon] = \{z \in \mathbb{C}_p / v(T - z) \geqslant \delta\}$ and $B(T, \varepsilon_j) = \{z \in \mathbb{C}_p / v(T - z) > \lambda_j\}$. It is clear that for z in the complementary of $B(T, \varepsilon_j)$ the function under the integral (5) is zero. This function is just $\delta - \lambda_j$ on the ball $B[T, \varepsilon]$, and equal to $v(z - T) - \lambda$ on $B(T, \varepsilon_j) - B[T, \varepsilon]$. Hence one has

$$\Delta_\delta(T) - \Delta_{\lambda_j}(T) = (\delta - \lambda_j)\, \mu_T(B[T, \varepsilon]) + \int_{B(T, \varepsilon_j) \backslash B[T, \varepsilon]} (v(z - T) - \lambda_j)\, d\mu_T.$$

The right hand side of this equality is a sum of positive terms. Thus the equality (4) becomes

$$w(f_n/f'_n) = \Delta_\delta(T) + \sum_j (\deg/P_j)(\delta - \lambda_j)\, \mu_T(B[T, \varepsilon]) + \int_{B(T, \varepsilon_j) \backslash B[T, \varepsilon]} (v(z - T) - \lambda_j)\, d\mu_T, \tag{6}$$

where the summation is taken over all j such that $\lambda_j < \delta$. We now estimate the integral which appears in (6). Let us denote

$$I_j = \int_{B(T, \varepsilon_j) \backslash B[T, \varepsilon]} (v(z - T) - \lambda_j)\, d\mu_T.$$

We shall complete the integral I_j in "polar coordinates." Precisely, we shall perform the integration on circles of radius r and then integrate over r, with r in $[\varepsilon, \varepsilon_j)$. We set

$$A = \{v(T - z);\, z \in C(T)\}.$$

For any real number η and any $a \in \mathbb{C}_p$ let us denote $C(a, \eta) = \{z \in \mathbb{C}_p;\, v(z - a) = \eta\}$. Then one has

$$\int_{B(T, \varepsilon_j) \backslash B[T - \varepsilon]} (v(z - T) - \delta_j)\, d\mu_T = \sum_{\substack{r \in A \\ \lambda_j \leqslant r \leqslant \delta}} \int_{C(T, r)} (r - \delta_j)\, d\mu_T$$

$$= \sum_{\substack{r \in A \\ \lambda_j < r \leqslant \delta}} (r - \lambda_j)\, \mu_T(C(T, r)).$$

In this way the equality (4) becomes

$$w(f_n/f'_n) = \Delta_\delta(T) + \left(\sum_{\lambda_j < \delta} \deg(P_j)(\delta - \delta_j) \right) \mu_T(B[T, \varepsilon])$$
$$+ \sum_{\delta_j < \delta} \deg(P_j) \sum_{\substack{r \in A \\ \lambda_j < r \leqslant \delta}} (r - \lambda_j)\, \mu_T(C(T, r)).$$

Now write the last sum of the right hand side, call it S, in the form

$$S = \sum_{\substack{r \in A \\ r \leqslant \delta}} \left(\sum_{\lambda_j < r} \deg(P_j)(r - \lambda_j) \right).$$

Since the map $z \mapsto v(T - z)$ defined on $C(T)$ is continuous, and since $C(T)$ is compact, it follows that A is an increasing sequence $r_1 < r_2 < \cdots$ which tends to infinity. Then the inner sum in the above form of S can be written as

$$\sum_{\lambda_j < r_k} \deg(P_j)(r_k - \lambda_j) = \sum_{1 \leqslant i \leqslant k} \sum_{r_{i-1} < \lambda_j \leqslant r_i} \deg(P_j)(r - \lambda_j)$$
$$\geqslant \sum_{1 \leqslant i < k} \sum_{r_{i-1} < \lambda_j \leqslant r_i} \deg(P_j)(r_k - r_i)$$

(here we assume $r_0 = -\infty$). Therefore one has

$$w(f_n/f'_n) \geqslant \sum_{r_k < \delta} \mu_T(C(T, r_k)) \sum_{i=1}^{k} (r_k - r_i) \sum_{\lambda_j \in (r_{i-1}, r_j]} \deg(P_j).$$

Denoting

$$D_i = \sum_{\lambda_j \in (r_{i-1}, r_i]} \deg(P_j)$$

we derive

$$w(f_n/f'_n) \geqslant \sum_{r_k < \delta} \mu_T(C(T, r_k)) \sum_{i < k} (r_k - r_i)\, D_i$$
$$\geqslant \tfrac{1}{2} \sum_{r_k < \delta} (r_k - r_{k-1})\, \mu_T(B[T, r_k]) \left(\sum_{i \leqslant k-1} D_i \right). \tag{7}$$

Here the last inequality holds true since $\mu_T(C(T, r_k)) \geqslant \frac{1}{2} \mu_T(B[T, r_k])$. Indeed, one has $B[T, r_k] = C(T, r_k) \cup B(T, r_k)$, and so $\mu_T(B[T, r_1]) = \mu_T(C(T, r_k)) + \mu_T(B(T, r_k))$. If $z_k \in C(T)$ is such that $v(T - z_k) = r_k$, then

$z_k \in C(T, r_k)$ and $V(z_k, r_k) \subset C(T, r_k)$. This means that $\mu_T(C(T, r_k)) \geqslant \mu_T(B(T, r_k))$, and so $\mu_T(B[T, r_k]) \leqslant 2\mu_T(C(T, r_k))$, as claimed. Furthermore, one has $\sum_{i \leqslant k-1} D_i = \sum_{i \geqslant 1} D_i - \sum_{i \geqslant k} D_i = \deg(f'_n) - \sum_{i \geqslant k} D_i$, and so we can write

$$\begin{aligned} 0 < \mu_T(B[T, r_k]) &\leqslant \mu_T(B[T, r_k]) \left(1 + \sum_{i \leqslant k-1} D_i\right) \\ &= \mu_T(B[T, r_k]) \left(1 + \deg(f'_n) - \sum_{i \geqslant k} D_i\right) \\ &= \mu_T(B[T, r_k]) \left[\deg(f_n) - \sum_{i \geqslant k} D_i\right]. \end{aligned}$$

Since $v(T - \alpha_n) \geqslant \delta > r_k$, by Corollary 3.7 it follows that $\deg(f_\alpha)\, \mu_T(B[T, r_k])$ is an integer. Analogously, for $i \geqslant k$ one has $D_i = \sum_{\lambda_j > r_k} \deg(P_j)$, and so the number $D_i \mu_T(B[T, r_k])$ is also an integer. This holds true for $i > k$. For $i = k$ one has $D_k = \sum_{\lambda_j \in (r_{k-1}, r_k]} \deg(P_j)$. As above one has $\mu_T(B[T, r_k]) = \mu_T(B(T, r_k)) + \mu_T(C(T, r_k))$. But $B(T, r_k) = B[T, r_{k-1}]$ and so $\deg(P_j)\, \mu_T(B(T, r_k))$ is an integer. Also, $C(T, r_k)$ is a (disjoint) union of balls of the form $B[z, \rho]$ with $z \in C(T)$ and $\rho \leqslant r_{k-1}$, hence $\deg(P_j)\, \mu_T(C(T, r_k))$ too is an integer. Finally we conclude that $\mu_T(B[T, r_k])(1 + \sum_{i \leqslant k-1} D_i)$ is a non-zero natural number. It follows that $\mu_T(B[T, r_k]) \sum_{i \leqslant k-1} D_i \geqslant 1 - \mu_T(B[T, r_k]) > \frac{1}{2}$ for r_k large enough. If one takes the sum over k in (7), one obtains

$$w(f_n(X)/f'_n(X)) \geqslant (1/4) \sum_{\lambda_0 < r_k < \delta} (r_k - r_{k-1}).$$

Hence if $\delta \to \infty$ then $w(f_n(X)/f'_n(X)) \to \infty$. This shows that condition (E) holds true for any T of the first kind and so the proof of Theorem 7.2 is finished.

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